

## ON BERRY'S CONJECTURES ABOUT THE STABLE ORDER IN PCF

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**ABSTRACT.** PCF is a sequential simply typed lambda calculus language. There is a unique order-extensional fully abstract cpo-model of PCF, built up from equivalence classes of terms. In 1979, Gérard Berry defined the stable order in this model and proved that the extensional and the stable order together form a bicpo. He made the following two conjectures:

1) “Extensional and stable order form not only a bicpo, but a bidomain.”

We refute this conjecture by showing that the stable order is not bounded complete, already for finitary PCF of second-order types.

2) “The stable order of the model has the syntactic order as its image: If  $a$  is less than  $b$  in the stable order of the model, for finite  $a$  and  $b$ , then there are normal form terms  $A$  and  $B$  with the semantics  $a$ , resp.  $b$ , such that  $A$  is less than  $B$  in the syntactic order.”

We give counter-examples to this conjecture, again in finitary PCF of second-order types, and also refute an improved conjecture: There seems to be no simple syntactic characterization of the stable order. But we show that Berry’s conjecture is true for unary PCF.

For the preliminaries, we explain the basic fully abstract semantics of PCF in the general setting of (not-necessarily complete) partial order models (f-models). And we restrict the syntax to “game terms”, with a graphical representation.

## 1. INTRODUCTION

PCF is a simple functional programming language, a call-by-name typed lambda calculus with integers and booleans as ground types, some simple sequential operations on the ground types, and a fixpoint combinator. The concept of PCF was formed by Dana Scott in 1969, see the historical document [22]. It is used as a prototypical programming language to explore the relationship between operational and denotational semantics, see the seminal paper of Gordon Plotkin [19].

The (operational) *observational preorder*  $M \sqsubseteq_{op} N$  of two terms (of equal type) is defined as: For all contexts  $C[\ ]$  of integer type, if  $C[M]$  reduces to the integer  $n$ , then  $C[N]$  also reduces to the same  $n$ . The denotational semantics (the model) assigns to every

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term  $M$  an element  $\llbracket M \rrbracket$  of a partial order  $(D, \sqsubseteq)$  (usually a complete partial order, cpo) as meaning. The model is said to be (*order*) *fully abstract* if the two orders coincide:  $M \sqsubseteq_{op} N \iff \llbracket M \rrbracket \sqsubseteq \llbracket N \rrbracket$ . The standard model of Scott domains and continuous functions is adequate (i.e. the direction  $\Leftarrow$  of the coincidence), but not fully abstract, because the semantic domains contain finite elements that are not expressible as terms, like the parallel or function. First Robin Milner [14] constructed in 1977 a unique fully abstract order-extensional cpo-model of PCF that can be built up from equivalence classes of terms by some ideal completion. The problem to construct a fully abstract model of PCF that does not use the syntax of terms (the “full abstraction problem”) was the driving force of the subsequent developments.

In 1979 Gérard Berry published his PhD thesis [4] with the translated title “Fully abstract and stable models of typed lambda-calculi”, which is the main basis of our work. In order to sort out functions like the parallel or from the semantic domains, to get “closer” to the fully abstract model, he gave the definition of stable function: A function  $f$  is *stable* if for the computation of some finite part of the output a deterministic minimal part of the input is needed. In the case that there are only finitely many elements smaller than a finite element, this definition is equivalent to the definition of a *conditionally multiplicative function*  $f$ : If  $a$  and  $b$  are compatible, then  $f(a \sqcap b) = fa \sqcap fb$ . To make the operation of functional application of stable functions itself stable, Berry had to replace the pointwise order of functions, the extensional order, by the new stable order: Two functions are in the *stable order*,  $f \leq g$ , if for all  $x \leq y$ :  $fx = fy \sqcap gx$ . This entails the pointwise order, but it demands in addition that  $g$  must not output some result for input  $x$  that  $f$  outputs only for greater  $y$ .

Now Berry had a model (of PCF) of stable functions with the stable order. But this model did not respect the old (pointwise) extensional order of the standard model and so had new unwanted elements not contained in the standard model. To get a proper subset of the standard model, he introduced bicpo models. A *bicpo* is a set with two orders, an extensional and a stable one, both forming cpos and being connected in some way. He augmented Milner’s fully abstract cpo model by the stable order and proved that it consists of bicpos and its functions are conditionally multiplicative. In section 3 we show in addition that its stable order forms stable bifinite domains and therefore its functions are also stable and can be represented by *traces*, i.e. sets of tokens (or events) like in [7]. E.g. the function  $\llbracket \lambda f. \text{if}(\text{zero}(f0)) \text{ then } 0 \text{ else } \perp \rrbracket$  can be represented by the trace consisting of the tokens  $\{0 \mapsto 0\} \mapsto 0$  and  $\{\perp \mapsto 0\} \mapsto 0$ . Functions are in the stable order,  $f \leq g$ , iff the trace of  $f$  is a subset of the trace of  $g$ .

In his thesis Berry made the following two conjectures that we refute:

- 1) “Extensional and stable order in the fully abstract cpo-model of PCF form not only a bicpo, but a bidomain.”

This would mean (among other things) that the stable order is bounded complete and distributive. We give counter-examples in finitary PCF of second-order types to this conjecture. The idea is that the stable lub of two stably bounded elements  $a$  and  $b$  may entail a new token that was not present in  $a$  or  $b$ . This new token must be used in the syntax to separate a subterm denoting  $a$  from a subterm denoting  $b$  that cannot be unified in a common term. Therefore distributivity is not fulfilled, stable lubs are not taken pointwise. And worse: There may be a choice between different new tokens to be entailed, then there is a choice between different minimal stable upper bounds of  $a$  and  $b$ , but there is no stable

lub. The minimal stable upper bounds are pairwise stably incompatible, and the extensional lub  $a \sqcup b$  is one of them.

**2)** The extensional order of the fully abstract model coincides with the (syntactic) observational preorder. This leads to the question: Is there a syntactic characterization also for the stable order? Berry made the conjecture:

“The stable order of the model has the syntactic order as its image:

If  $a \leq b$  in the stable order, for finite  $a$  and  $b$ , then there are normal form terms  $A$  and  $B$  with  $\llbracket A \rrbracket = a$  and  $\llbracket B \rrbracket = b$ , such that  $A \prec B$  in the syntactic order.”

Berry proved the converse direction: If  $A \prec B$ , then  $\llbracket A \rrbracket \leq \llbracket B \rrbracket$ , and proved the conjecture for first-order types.

Our simplest counter-example to this conjecture is a situation of four terms  $A \prec B \cong C \prec D$ , where  $\cong$  is observational equivalence, so that  $\llbracket A \rrbracket \leq \llbracket D \rrbracket$ , but there is no way to find terms  $A' \cong A$ ,  $D' \cong D$  with  $A' \prec D'$ . The elimination of some token of  $D$  depends on the prior elimination of some other token, so that two  $\prec$ -steps are necessary to get from  $D$  down to  $A$ .

We further give examples where such a chain of  $\prec$ -steps (with intermediate  $\cong$ -steps) of any length is necessary. This proposes an improved conjecture, the “chain conjecture”: Instead of  $A \prec B$  we demand the existence of a chain between  $A$  and  $B$ . But we also refute this conjecture. Although stable order and syntactic order are connected, there seems to be no simple syntactic characterization of the stable order in PCF.

All our counter-examples for both conjectures are in finitary PCF of second-order types. They all share a common basic idea: There is a token  $t$  of the form

$$t = \{\perp 0 \mapsto 0, \perp 1 \mapsto 1\} \mapsto 0,$$

which can be expressed by the term

$$M = \lambda g. \text{if zero}(g \perp 0) \text{ then } (\text{if zero}(g \perp 1) \text{ then } \perp \text{ else } 0) \text{ else } \perp.$$

$t$  is also contained in the trace of the term

$$N = \lambda g. \text{if zero}(g \perp (\text{if zero}(g \perp 1) \text{ then } \perp \text{ else } 0)) \text{ then } 0 \text{ else } \perp$$

which nests the two function calls of  $g$  and which contains in addition the token  $s = \{\perp \perp \mapsto 0\} \mapsto 0$ . Besides  $t$  there may be other tokens in the trace of a term (they demand the first argument of  $g$ ) which inhibit the flat, unnested form  $M$  and “force” the nested form  $N$ , which entails the token  $s$ .

The necessary ingredients for the counter-examples are: at least second-order type with some functional parameter of arity at least 2, at least two different ground values 0 and 1, and the need for nested function calls.

If we restrict the calculus to a single ground value 0, we get unary PCF, and in this case both of Berry's conjectures are true: The fully abstract model is a bidomain, in fact it is the standard semantical bidomain construction, proved by Jim Laird in [10]. And we prove that the syntactic order is the image of the stable order, using Laird's proof that every type in unary PCF is a definable retract of some first-order type.

The need for nested function calls is the result of a “restriction” of PCF: There is no operator to test if a function demands a certain argument, so that this information could be used in an if-then-else. Jim Laird has shown that in a language with such control operators (SPCF) nested function calls can be eliminated, and also every type of SPCF is a definable

retract of a first-order type [11]. Therefore I am convinced, though I do not prove it here, that also for SPCF the syntactic order is the image of the stable order.

The above mentioned “restriction” of PCF is generally the reason for many irregularities of the semantics of PCF and the difficulty of the full abstraction problem. An important result is the undecidability of finitary PCF [12]. This means that the observational equivalence of two terms of finitary PCF is undecidable, and also the question whether there is a term for a functional value table. As remarked in the introduction to [7], this result restricts the possible fully abstract models of PCF to be not “finitary” in some sense. There have been several solutions for semantical fully abstract models of PCF: A model of continuous functions restricted by Kripke logical relations [18], and game semantics [1, 9, 16]. In game semantics a term of PCF is modeled by a strategy of a game, i.e. by a process that performs a dialogue of questions and answers with the environment, the opponent. These strategies are still intensional; the fully abstract model is formed by a quotient, the extensional collapse. The strategies can be identified with PCF Böhm trees of a certain normal form, see also [2, section 6.6]. We call these Böhm trees “game terms” and prove that it is sufficient to formulate all our results in the realm of game terms, esp. that if two terms are syntactically ordered, then there are equivalent game terms so ordered. This simplifies the proofs of the counter-examples. We also introduce a graphical notation for game terms that facilitates the handling of larger examples.

It was an open problem whether the game model is isomorphic to Milner’s fully abstract cpo-model, i.e. whether its domains are cpos. This problem was solved by Dag Normann [17]: Its domains are not cpos, i.e. there are directed sets that have no lub. Then Vladimir Sazonov made a first attempt to build a general theory for these non-cpo domains [20, 21]. His main insight was that functions are continuous only with respect to certain lubs of directed sets that he calls “natural lubs”; these are the hereditarily pointwise lubs.

We want to place our results in the context of these new, more general models. For the semantic preliminaries we give a simple definition of a set of well-behaved (not-necessarily complete) partial order fully abstract models of PCF: These *f-models* are sets of ideals of finite elements, such that application is defined and every PCF-term has a denotation. Sazonov’s natural lubs correspond to our *f-lubs*, which are defined with respect to the finite elements.

I found the counter-example to Berry’s second conjecture around the year 1990, but did not yet publish it. As far as I know, nobody else tackled Berry’s problems. The reason for this seems to be that they were simply forgotten. The stable order in the fully abstract model was never explored after Berry; a reason may be that he never prepared a journal version of his thesis, which is not easily accessible. The recommended introduction to our subject is the report “Full abstraction for sequential languages: The state of the art” [5], which contains the thesis in condensed form, but lacks most proofs. There is also an article [3] published by Berry before his thesis, which is not recommended, because section 4.5 (bidomains) is wrong (different definition of bidomain, the first conjecture is stated as theorem). An excellent general introduction to domains, stability and PCF (and many other things) is the textbook [2]. But for the stable order in the fully abstract model of PCF the only detailed source remains Berry’s thesis.

Here is the structure of the paper. The counter-examples are given in the order of their discovery, i.e. in the order of increasing complexity.

2. Syntax of PCF.
3. Semantics of PCF: non-complete partial order f-models:  
We introduce f-models as general (not-necessarily complete) partial order fully abstract models of PCF and give the properties of the stable order in this general context. (The normal fully abstract cpo-model of PCF is a special case.)
4. Game terms:  
We describe the construction of game terms by the finite projections and give a graphical notation for game terms.  
The expert who is interested only in the counter-examples may skip the introductory sections 2-4; reading only the definition of game terms and their graphical notation at the beginning of section 4.
5. The syntactic order is not the image of the stable order:  
We prove Berry's second conjecture for first-order types, give a counter-example in a second-order type (a chain of length 2), prove the existence of chains of any least length, and explore universal terms.
6. The stable order is not bounded complete: no bidomain:  
We prove Berry's first conjecture for first-order types. In a second-order type we give an example of a stable lub that does not fulfill distributivity, and an example of two stably bounded elements without stable lub.
7. Refutation of the chain-conjecture:  
We refute the improved second conjecture that the stable order entails a chain of terms.
8. Unary PCF:  
We prove Berry's second conjecture for unary PCF, with the aid of Jim Laird's definable retractions from any type to some first-order type [10].
9. Outlook.

## 2. SYNTAX OF PCF

In this section we give the syntactic definitions of PCF [19, 5, 2]. The programming language PCF is a simply typed lambda calculus with arithmetic and fixpoint operators. It usually comes with two ground types  $\iota$  (integers) and  $o$  (booleans). We simplify the language and use only the ground type  $\iota$  (integers); the booleans are superfluous and can be coded as integers, the intensional structure of the terms stays the same.

The **types** are formed by  $\iota$  and function types  $\sigma \rightarrow \tau$  for types  $\sigma$  and  $\tau$ .

The typed **constants** are:

$0, 1, 2, \dots : \iota$ , the integers;

**suc, pre**:  $\iota \rightarrow \iota$ , successor and predecessor function;

**if \_ then \_ else \_**:  $\iota \rightarrow \iota \rightarrow \iota \rightarrow \iota$ , this conditional tests if the first argument is 0.

(We write e.g. **if**  $x$  **then**  $y$  for the application of this function to only two arguments.)

The PCF **terms** comprise the constants and the typed constructs by the following rules:

$\perp^\sigma : \sigma$  for any type  $\sigma$ , the undefined term.

$x^\sigma : \sigma$  for any variable  $x^\sigma$ .

If  $M : \tau$ , then  $\lambda x^\sigma.M : \sigma \rightarrow \tau$ , lambda abstraction.

If  $M : \sigma \rightarrow \tau$  and  $N : \sigma$ , then  $MN : \tau$ , function application.

If  $M : \sigma \rightarrow \sigma$ , then  $YM : \sigma$ ,  $Y$  is the fixpoint operator.

$\text{PCF}^\sigma$  is the set of all PCF terms of type  $\sigma$ , and  $\text{PCF}_c^\sigma$  is the set of the closed terms of these.

Type annotations of  $\perp$  and of variables will often be omitted.

We use the (semantic) symbol  $\perp$  also as syntactic term, instead of the usual  $\Omega$ .

We define the *syntactic order*  $\prec$  ( $\perp$ -match prefix order) on terms of the same type:

$M \prec N$  iff  $N$  can be obtained by replacing some occurrences of  $\perp$  in  $M$  by terms.

The **reduction rules** are (where  $n$  is a variable for integer constants):

$(\lambda x.M)N \rightarrow M[x := N]$ , the usual  $\beta$ -reduction;

$YM \rightarrow M(YM)$ ;

$\text{succ } n \rightarrow (n + 1)$ ;

$\text{pre } n \rightarrow (n - 1)$ , for  $n \geq 1$ ;

if 0 then  $M$  else  $N \rightarrow M$ ;

if  $n$  then  $M$  else  $N \rightarrow N$ , for  $n \geq 1$ .

The *reduction relation*  $\rightarrow$  is one step of reduction by these rules in any term context.

It is confluent.  $\rightarrow^*$  is the reflexive, transitive closure of  $\rightarrow$ .

A *program* is a closed term of type  $\iota$ .

The *operational (observational) preorder*  $\sqsubseteq_{op}$  on terms of the same type is defined as:

$M \sqsubseteq_{op} N$  ( $M$  is *operationally less defined than*  $N$ ) iff

$P[M] \rightarrow^* n$  implies  $P[N] \rightarrow^* n$  for all contexts  $P[\ ]$  such that  $P[M]$  and  $P[N]$  are both programs.

The *operational equivalence* is defined as:  $M \cong N$  iff  $M \sqsubseteq_{op} N$  and  $N \sqsubseteq_{op} M$ .

### 3. SEMANTICS OF PCF: NON-COMPLETE PARTIAL ORDER F-MODELS

This section gives an exposition of the fully abstract semantics of PCF with the stable order, as far as it is needed to understand the results of this paper. The proofs are omitted, as they are easy and/or already known in some form.

The order-extensional fully abstract cpo-model of PCF was first constructed by Robin Milner [14] based on terms of an SKI-combinator calculus. Later Gérard Berry's thesis [4] constructed this model based on the proper  $\lambda$ -terms. This model is the ideal completion of the finite elements; every directed set has a lub.

Then came the fully abstract game models of PCF [1, 9, 16]. The elements of these models can be represented by the (infinite) Böhm trees of PCF. It was an open problem whether the game model is isomorphic to Milner's model, i.e. whether its domains are cpos.

This problem was solved by Dag Normann [17]: Its domains are not cpos, i.e. there are directed sets that have no lub. Then Vladimir Sazonov made a first attempt to build a general theory for these non-cpo domains [20, 21]. His main insight was that functions are continuous only with respect to certain lubs of directed sets that he calls "natural lubs"; these are the hereditarily pointwise lubs.

We want to place our results in the context of these new, more general models. Therefore we give a simple definition of a set of well-behaved (not-necessarily complete) partial order fully abstract models of PCF: These *f-models* are sets of ideals of finite elements, such that application is defined and every PCF-term has a denotation. Sazonov's natural lubs correspond to our *f-lubs*, which are defined with respect to the finite elements.

We state the usual properties for these f-models; the essence of their proofs is already contained in Berry's construction. Our aim is the definition of the stable order and of conditionally multiplicative (cm) functions. All functions in f-models are cm. We can further show, in addition to Berry, that the domains have property I under the stable order and therefore the functions are stable and we can work with their traces.

We need the following PCF terms, the *finite projections* on type  $\sigma$  of *grade*  $i$ ,  $\Psi_i^\sigma : \sigma \rightarrow \sigma$ :

$$\begin{aligned}\Psi_i^\ell &= \lambda x^\ell. \text{if } x \text{ then } 0 \text{ else if } \text{pre}^1 x \text{ then } 1 \text{ else } \dots \text{if } \text{pre}^i x \text{ then } i \text{ else } \perp \\ \Psi_i^{\sigma \rightarrow \tau} &= \lambda f^{\sigma \rightarrow \tau}. \lambda x^\sigma. \Psi_i^\tau(f(\Psi_i^\sigma x))\end{aligned}$$

We also need the following terms for the glb functions on all types,  $\text{inf}^\sigma : \sigma \rightarrow \sigma \rightarrow \sigma$ , here in a liberal syntax:

$$\begin{aligned}\text{inf}^\ell &= \lambda x^\ell y^\ell. \text{if } x = y \text{ then } x \text{ else } \perp \\ &= \lambda x^\ell y^\ell. \text{if } x \text{ then if } y \text{ then } 0 \text{ else } \perp \\ &\quad \text{else } \text{suc}(\text{inf}^\ell(\text{pre } x)(\text{pre } y)) \\ \text{inf}^{\sigma \rightarrow \tau} &= \lambda f^{\sigma \rightarrow \tau} g^{\sigma \rightarrow \tau}. \lambda x^\sigma. \text{inf}^\tau(fx)(gx)\end{aligned}$$

When applied to a closed term  $M : \sigma$ , the function term  $\Psi_i^\sigma$  serves as a “filter” that lets only pass integer values  $\leq i$  as input or output to  $M$ . This serves to define the finite elements of the intended model.

**Definition 3.1.** A term  $M : \sigma$  is a *finite term of grade*  $i$  if it is closed and  $M \cong \Psi_i^\sigma M$ .  $\mathcal{F}_i^\sigma = \{ [\Psi_i^\sigma M]_{op} \mid M \in \text{PCF}_c^\sigma \}$  is the set of *finite elements of grade*  $i$  of type  $\sigma$ , where  $[X]_{op}$  is the equivalence class of term  $X$  under the operational equivalence  $\cong$ .  $\mathcal{F}^\sigma = \bigcup_i \mathcal{F}_i^\sigma$  is the set of *finite elements* of type  $\sigma$ .

The finite elements are partially ordered by the extension of the operational preorder  $\sqsubseteq_{op}$  to equivalence classes.

An *ideal* of finite elements of type  $\sigma$  is a set  $S \subseteq \mathcal{F}^\sigma$  such that:  $S \neq \emptyset$  and  $a, b \in S \implies \exists c \in S. a \sqsubseteq_{op} c$  and  $b \sqsubseteq_{op} c$ , and  $a \in S, b \in \mathcal{F}^\sigma$  and  $b \sqsubseteq_{op} a \implies b \in S$ .

$I(\mathcal{F}^\sigma)$  is the set of ideals of finite elements of type  $\sigma$ .

$a \in \mathcal{F}^\sigma$  is identified with the ideal  $\downarrow\{a\}$ , the downward closure of  $\{a\}$ .

There is an operation *apply* on ideals of finite elements. For  $f \in I(\mathcal{F}^{\sigma \rightarrow \tau})$ ,  $d \in I(\mathcal{F}^\tau)$ :

$$\text{apply}(f, d) = \downarrow\{f'd' \mid f' \in f, d' \in d\} \in I(\mathcal{F}^\tau),$$

where  $f'd' = [MN]_{op}$  for  $M \in f'$ ,  $N \in d'$ .  $\text{apply}(f, d)$  is simply written  $fd$ .

**Definition 3.2.** An *f-model* of PCF (“f” means: based on finite elements) is a collection of  $D^\sigma \subseteq I(\mathcal{F}^\sigma)$  for every type  $\sigma$ , each  $D^\sigma$  ordered by the subset relation written  $\sqsubseteq$ , such that for  $f \in D^{\sigma \rightarrow \tau}$ ,  $d \in D^\tau$ :  $fd \in D^\tau$ , and such that every closed term  $M : \sigma$  has its denotation in  $D^\sigma$ :  $\downarrow\{[\Psi_i^\sigma M]_{op} \mid i \geq 0\} \in D^\sigma$ . The lubs w.r.t.  $\sqsubseteq$  will be written  $\sqcup$  and  $\bigsqcup$ , the glbs  $\sqcap$  and  $\bigsqcap$ .

All f-models coincide on their part of the finite elements w.r.t. both extensional  $\sqsubseteq$  and stable  $\leq$  order. In the following sections, propositions will mostly deal with finite elements. The propositions are valid for all f-models if not otherwise stated.

To every f-model we can associate the *semantic map*  $\llbracket \cdot \rrbracket : \text{PCF}^\sigma \rightarrow \text{ENV} \rightarrow D^\sigma$ , where ENV is the set of environments  $\rho$  that map every variable  $x^\sigma$  to some  $\rho(x^\sigma) \in D^\sigma$ . If  $M : \sigma$  is a term with the free variables  $x_1, \dots, x_n$ , then

$$\llbracket M \rrbracket \rho = \downarrow \{ [\Psi_i^\sigma M[x_1 := N_1, \dots, x_n := N_n]]_{op} \mid i \geq 0, [N_j]_{op} \in \rho(x_j) \}.$$

For closed terms  $M$  we also write  $\llbracket M \rrbracket$  for  $\llbracket M \rrbracket \perp$ .

There are three outstanding examples of f-models: There is the least f-model that consists of just the ideals denoting closed PCF-terms. There is the greatest f-model consisting of all ideals; this is Milner's and Berry's cpo-model. And there is the game model consisting of all denotations of (infinite) PCF-Böhm-trees. By Normann's result [17] we know that the game model is properly between the least and the greatest f-models.

Now we will collect the most important properties of f-models. In the following the  $D^\sigma$  are the domains of some f-model.

**Lemma 3.3.** *Every  $\mathcal{F}_i^\sigma$  has finitely many elements.*

*The semantics of the  $\text{inf}^\sigma$ -terms are the glb-functions with respect to the order  $\sqsubseteq$ ; we write  $\sqcap$  for these functions.*

*If  $d, e \in \mathcal{F}_i^\sigma$ , then  $d \sqcap e \in \mathcal{F}_i^\sigma$ .*

*If  $d, e \in \mathcal{F}_i^\sigma$  are compatible (bounded), i.e. there is some  $a \in D^\sigma$  with  $d \sqsubseteq a$  and  $e \sqsubseteq a$ , then there is a lub  $d \sqcup e \in \mathcal{F}_i^\sigma$ .*

With this lemma we can prove:

**Proposition 3.4.** *All  $D^{\sigma \rightarrow \tau}$  are order-extensional, i.e. :*

$$\begin{aligned} \text{If } f, g \in D^{\sigma \rightarrow \tau}, \text{ then } f \sqsubseteq g &\iff \forall d \in D^\sigma. fd \sqsubseteq gd \\ &\iff \forall d \in \mathcal{F}^\sigma. fd \sqsubseteq gd \end{aligned}$$

Elements of  $D^{\sigma \rightarrow \tau}$  will be identified with the corresponding functions. apply and these functions are all monotone. They are continuous with respect to certain directed lubs, the f-lubs.

**Definition 3.5.** The directed set  $S \subseteq D^\sigma$  has the f-lub  $s$ , written  $S \rightarrow s$ , iff  $s$  is an upper bound of  $S$  and for all finite  $x \sqsubseteq s$  there is some  $y \in S$  with  $x \sqsubseteq y$ . (This is equivalent to:  $s$  is the set-theoretical union of  $S$ .  $s$  is also the lub of  $S$  w.r.t.  $\sqsubseteq$ .)

A function  $f : D^\sigma \rightarrow D^\tau$  is *f-continuous*, iff it is monotone and respects f-lubs of directed sets  $S \subseteq D^\sigma$ , i.e. if  $S \rightarrow s$ , then  $fS \rightarrow fs$ . (With  $fS = \{fx \mid x \in S\}$ .)

**Proposition 3.6.** *The apply operation is f-continuous on the domain  $D^{\sigma \rightarrow \tau} \times D^\sigma$ . (With component-wise order and pairs of finite elements as finite elements.) Therefore apply is f-continuous in each argument, and the functions of  $D^{\sigma \rightarrow \tau}$  are f-continuous.*

Please note that there may be lubs of directed sets that are not f-lubs; and that there may be finite elements that are not compact in the usual sense with respect to general directed lubs.

The f-lubs are exactly the directed lubs for which all functions are continuous: If we have a directed lub that is not an f-lub, then this lub contains a finite element that is not contained in the directed set. The PCF-function that “observes” (or “tests”) this finite element is a function that is not continuous for the directed set.



In the greatest f-model all lubs of directed sets are f-lubs. If  $S \rightarrow s$  in the greatest f-model, then the same holds in all f-models that contain  $s$  and the elements of  $S$ . Directed f-lubs coincide with the directed hereditarily pointwise lubs, the natural lubs of Sazonov.

**Proposition 3.7.** *The semantic map of an f-model fulfills the usual equations, i.e. the constants have their intended meanings, and:*

$$\begin{aligned}\llbracket \lambda x.M \rrbracket \rho d &= \llbracket M \rrbracket \rho[x := d] \\ \llbracket MN \rrbracket \rho &= \llbracket M \rrbracket \rho \llbracket N \rrbracket \rho \\ \llbracket Y M \rrbracket \rho &= \bigsqcup_n ((\llbracket M \rrbracket \rho)^n \perp)\end{aligned}$$

**Proposition 3.8** (Berry, 3.6.11 in [4]). *Define the functions  $\psi_i^\sigma = \llbracket \Psi_i^\sigma \rrbracket \perp : D^\sigma \rightarrow D^\sigma$ . For all  $\sigma$ ,  $(\psi_i^\sigma)$  is an increasing sequence of finite projections with f-lub the identity id:*

$$\begin{aligned}\psi_i^\sigma &\sqsubseteq \text{id} \\ \psi_i^\sigma \circ \psi_i^\sigma &= \psi_i^\sigma \\ \psi_i^\sigma &\sqsubseteq \psi_{i+1}^\sigma \\ \{\psi_i^\sigma\} &\rightarrow \text{id} \\ \psi_i^\sigma(D^\sigma) &= \mathcal{F}_i^\sigma\end{aligned}$$

**Proposition 3.9.** *Every f-model is fully abstract: For all terms  $M, N$  of the same type  $(\forall \rho \in \text{ENV}. \llbracket M \rrbracket \rho \sqsubseteq \llbracket N \rrbracket \rho) \iff M \sqsubseteq_{op} N$ .*

In the rest of this section we will define the stable order in f-models and collect the corresponding properties that will be needed in this paper.

The definition of the stable order  $\leq$  is given by Berry [4, 4.8.6, page 4-93] for the fully abstract cpo-model as follows:

$$\begin{aligned}\text{For } d, e \in D^\iota : d \leq e &\iff d \sqsubseteq e \\ \text{For } f, g \in D^{\sigma \rightarrow \tau} : f \leq g &\iff \forall x \in D^\sigma. fx \leq gx \text{ and} \\ &\quad \forall x, y \in D^\sigma. x \uparrow_\leq y \implies fx \sqcap gy = fy \sqcap gx\end{aligned}$$

(Here  $\uparrow_\leq$  means compatibility w.r.t.  $\leq$ .)

This definition serves as well for our f-models, but I prefer the equivalent (w.r.t. the full type hierarchy) form:

**Definition 3.10** (stable order  $\leq$ ).

$$\begin{aligned}\text{For } d, e \in D^\iota : d \leq e &\iff d \sqsubseteq e \\ \text{For } f, g \in D^{\sigma \rightarrow \tau} : f \leq g &\iff \forall x, y \in \mathcal{F}^\sigma. x \leq y \implies fx = fy \sqcap gx\end{aligned}$$

The order  $\leq$  is extended pointwise to environments from ENV, here used in the definition of  $\leq$  on denotations:

$$\text{For } f, g \in \text{ENV} \rightarrow D^\sigma : f \leq g \iff \forall \rho, \varepsilon \in \text{ENV}. \rho \leq \varepsilon \implies f\rho = f\varepsilon \sqcap g\rho$$

The lubs w.r.t.  $\leq$  will be written  $\vee$  and  $\bigvee$ , the glbs  $\wedge$ .

Note that  $\sqcap$  is by definition the glb w.r.t. the *extensional* order  $\sqsubseteq$ . But we can prove the following:

**Proposition 3.11.** *In any actual  $f$ -model the following holds:*

*For  $f, g \in D^\sigma$ : If  $f, g$  are  $\leq$ -compatible in the greatest  $f$ -model, then  $f \sqcap g$  is also the glb w.r.t.  $\leq$ . (Note: If  $f, g$  are  $\leq$ -compatible in the actual  $f$ -model, then they are also compatible in the greatest  $f$ -model.)*

*If  $f \leq g$  then  $f \sqsubseteq g$ .  $\leq$  is a partial order on  $D^\sigma$ .*

$$\begin{aligned} \text{For } f, g \in D^{\sigma \rightarrow \tau} : f \leq g &\iff \forall x \in D^\sigma. fx \leq gx \text{ and} \\ &\forall x, y \in D^\sigma. x \leq y \implies fx = fy \sqcap gx \end{aligned}$$

*The definition of  $\leq$  can be given in “uncurried” form with vectors of arguments, the order  $\leq$  extended componentwise:*

$$\begin{aligned} \text{For } f, g \in D^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota} : f \leq g &\iff \forall x_1, y_1 \in D^{\sigma_1}, \dots, x_n, y_n \in D^{\sigma_n}. \\ &(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \implies \\ &fx_1 \dots x_n = fy_1 \dots y_n \sqcap gx_1 \dots x_n \end{aligned}$$

*Proof.* The proof that  $f \sqcap g$  is the glb w.r.t.  $\leq$  (for  $\leq$ -compatible  $f, g$ ) is by induction on the type  $\sigma$ . It uses only the definition of  $\leq$  and that  $\sqcap$  is the glb w.r.t.  $\sqsubseteq$ , no stability (or conditional multiplicativity) is used.  $\square$

**Definition 3.12.**  $f \in D^{\sigma \rightarrow \tau}$  is *conditionally multiplicative (cm)* if

$$\forall x, y \in \mathcal{F}^\sigma. x \uparrow_\leq y \implies f(x \sqcap y) = fx \sqcap fy$$

Analogously for denotations  $f \in \text{ENV} \rightarrow D^\sigma$ .

This definition can also be given in “uncurried” form:  $f \in D^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota}$  is cm iff

$$\begin{aligned} \forall x_1, y_1 \in D^{\sigma_1}, \dots, x_n, y_n \in D^{\sigma_n}. (x_1, \dots, x_n) \uparrow_\leq (y_1, \dots, y_n) &\implies \\ f(x_1 \sqcap y_1) \dots (x_n \sqcap y_n) &= fx_1 \dots x_n \sqcap fy_1 \dots y_n \end{aligned}$$

**Theorem 3.13** (Berry, 4.8.10 in [4]). *In an  $f$ -model, all functions from domains  $D^{\sigma \rightarrow \tau}$  are cm. All denotations  $\llbracket M \rrbracket$  are cm.*

*Proof.* Berry first proves the property cm for the denotations of normal form terms by induction on the size of the type. Then it is extended to all functions by continuity.  $\square$

**Proposition 3.14** (Berry [4], syntactic monotony w.r.t.  $\leq$ ).

*For every context  $C[\ ]$  with hole of type  $\sigma$ , and terms  $M, N : \sigma$ :*

*If  $\llbracket M \rrbracket \leq \llbracket N \rrbracket$  then  $\llbracket C[M] \rrbracket \leq \llbracket C[N] \rrbracket$ .*

*Therefore, for terms  $M, N : \sigma$ : If  $M \prec N$  then  $\llbracket M \rrbracket \leq \llbracket N \rrbracket$ .*

We will also write  $M \leq N$  for  $\llbracket M \rrbracket \leq \llbracket N \rrbracket$ .

Now we show property I of  $(D^\sigma, \leq)$  and the representation of all functions by traces, which is not contained in Berry’s thesis.

**Proposition 3.15.** *For the finite projections we have:  $\psi_i^\sigma \leq \psi_{i+1}^\sigma$  and  $\psi_i^\sigma \leq \text{id}$ .*

*The  $\mathcal{F}_i^\sigma$  are downward closed w.r.t.  $\leq$ : If  $d \in \mathcal{F}_i^\sigma$ ,  $e \in D^\sigma$  and  $e \leq d$ , then  $e \in \mathcal{F}_i^\sigma$ .*

*Therefore the domains  $(D^\sigma, \leq)$  have the property I: There are only finitely many elements under each finite element.*

*Proof.* The proof of  $\psi_i^\sigma \leq \text{id}$  is by induction on the type  $\sigma$ ; the induction step is in the proof of proposition 12.4.4 in the section on stable bifinite domains of [2, page 287]. The downward closedness of  $\mathcal{F}_i^\sigma$  is an easy consequence and can be found at the same place.  $\square$

Because of property I, all our functions of  $D^{\sigma \rightarrow \tau}$  (which are cm) are also stable, and therefore can be represented by traces. We chose the trace of the uncurried form.

**Definition 3.16.** Let  $f \in D^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota}$ ,  $n \geq 0$ ,  $x_i \in D^{\sigma_i}$  and  $f x_1 \dots x_n = j$  for some integer  $j$ .

Then there are  $y_i \in \mathcal{F}^{\sigma_i}$ ,  $y_i \leq x_i$ , with  $f y_1 \dots y_n = j$  and  $(y_1, \dots, y_n)$  is the  $\leq$ -least vector with this property. (This is the meaning of:  $f$  is stable.)

In this case we say that  $y_1 \mapsto \dots \mapsto y_n \mapsto j$  is a *token* of  $f$ .

The set of all tokens of  $f$  is called the *trace* of  $f$ , written  $\mathcal{T}(f)$ .

The  $y_i$  in the token will be represented by traces again. We will use a liberal syntax for tokens and traces, writing  $\perp$  for the trace  $\emptyset$ ,  $0$  for the trace  $\{0\}$  of  $0$ , and also  $00 \mapsto 0$  for the token  $\{0\} \mapsto \{0\} \mapsto 0$ . If  $M$  is a closed term, we write simply  $\mathcal{T}[\![M]\!]$  for the trace of its denotation  $\llbracket M \rrbracket \perp$ .

**Proposition 3.17.** For  $f, g \in D^\sigma$ :  $f \leq g$  iff  $\mathcal{T}(f) \subseteq \mathcal{T}(g)$ .

If  $f, g$  are  $\leq$ -compatible in the greatest  $f$ -model, then  $\mathcal{T}(f \sqcap g) = \mathcal{T}(f) \cap \mathcal{T}(g)$ .

$f \in D^\sigma$  is finite of grade  $i$ ,  $f \in \mathcal{F}_i^\sigma$ , iff all numbers in the trace of  $f$  are  $\leq i$ .

#### 4. GAME TERMS

Berry's conjectures demand the existence of certain finite PCF-terms. In this section we show that we may restrict these finite terms to terms in a certain standard normal form that we call *game terms*. This will simplify the proofs of the counter-examples, and is also an interesting result itself. Game terms first appeared in the literature on game semantics as terms representing game strategies; in [1, section 3.2] they were called (finite and infinite) “evaluation trees”, in [9, section 7.3] “finite canonical forms” that correspond to compact innocent strategies, and in [2, section 6.6] “PCF Böhm trees”. The textbook article on “PCF Böhm trees” comes closest to our approach, as it introduces a semantics in the form of Böhm trees and has to solve similar problems in the needed syntactic transformations. But we do not employ a (game or other) semantics, i.e. we do not interpret the PCF-constants by infinite strategies or Böhm trees; our approach is purely syntactic. We take a finite PCF-term, apply an operator that resembles the finite projection  $\Psi_i^\sigma$  and reduce the resulting term to its game term form. We show that the transforming reductions respect the syntactic order  $\prec$  (used in the refutation of Berry's second conjecture), and this will also enable us to proceed to infinite game terms. We also introduce a graphical representation of game terms that makes the behaviour of terms better visible.

First we introduce an additional new construct for the PCF language, for every  $i \geq 0$ : If  $M, N_0, \dots, N_i : \iota$ , then  $\text{case}_i M N_0 \dots N_i : \iota$ .

Please note that  $\text{case}_i$  is not a constant, but the whole case-expression is a new construct of the language, it is no application. We call the new terms (PCF-)case-terms, and a case-term with all case-expressions as  $\text{case}_i$  for fixed  $i$  we call  $\text{case}_i$ -term. The reduction rule for  $\text{case}_i$  is:

$$\text{case}_i n N_0 \dots N_i \rightarrow N_n, \text{ for } 0 \leq n \leq i$$

The case-expression is equivalent to a PCF-term:

$$\text{case}_i M N_0 \dots N_i \cong \text{if } M \text{ then } N_0 \text{ else if } \text{pre}^1 M \text{ then } N_1 \text{ else } \dots \text{ if } \text{pre}^i M \text{ then } N_i \text{ else } \perp$$

This is the “filter” as it appears in the finite projection term  $\Psi_i^t$ . So  $\text{case}_i$  does not enhance the expressiveness of PCF. It is merely a “macro” that is used as short expression for the filter term above, to keep the unity of the filter term in the transformation to game terms.

The syntactic order  $\prec$  is defined on case-terms as follows:

$$\text{case}_i M N_0 \dots N_i \prec \text{case}_j M' N'_0 \dots N'_j \text{ iff } i \leq j, M \prec M' \text{ and } N_k \prec N'_k \text{ for } 0 \leq k \leq i.$$

This is equivalent to the syntactic order on the macro expansions of the case-expressions.

**Definition 4.1.** *Game terms* are the well-typed PCF-case-terms that are furthermore produced by the following grammar:

$$\begin{aligned} M, N ::= & \perp^\sigma, \sigma \text{ any type} \\ & \lambda x_1 \dots x_n. m, m \text{ integer constant, } n \geq 0 \\ & \lambda x_1 \dots x_n. \text{case}_i(y M_1 \dots M_m) N_0 \dots N_i, n, m, i \geq 0 \end{aligned}$$

Please note that  $\lambda x_1 \dots x_n.$  vanishes for  $n = 0$ , so needed for the  $N_k$  of type  $\iota$ .

A *game term of grade  $i$* ,  $i \geq 0$ , is a game term that is a  $\text{case}_i$ -term (every  $\text{case}$  is  $\text{case}_i$ ) with all integer constants  $\leq i$ . (This entails that a closed game term of grade  $i$  is a finite term of grade  $i$ .)

We suppose some implicit notion of position in terms and define the *output positions* of a game term recursively:

The *output position* of a game term  $\lambda x_1 \dots x_n. m$  ( $m$  integer constant) is the position of the  $m$ , its *value* is  $m$ .

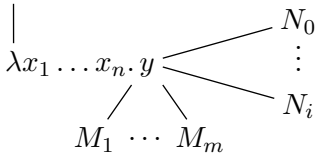
The *output positions* of a game term  $\lambda x_1 \dots x_n. \text{case}_i(y M_1 \dots M_m) N_0 \dots N_i$  are the output positions in the subterms  $N_0, \dots, N_i$ .

The output positions give the positions where integer output at the top level occurs. We denote replacements of the values of output positions in brackets  $[\dots]$ ; e.g.  $M[n := A_n]$  means the replacement of all values  $n$  of output positions in  $M$  by the corresponding term  $A_n$ .

A *game term of pregrade  $i$* ,  $i \geq 0$ , is a game term that is a  $\text{case}_i$ -term and for all integer constants  $m$  that are *not* in output position it is  $m \leq i$ . (So the values of output positions are not restricted.)

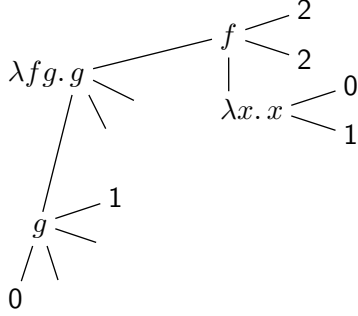
We will use a graphical representation of game terms in the next sections:

A subterm  $\lambda x_1 \dots x_n. \text{case}_i(y M_1 \dots M_m) N_0 \dots N_i$  is represented in the graph by a node of the form:



The upper parent of this node is connected to the  $\lambda$ ; if the  $\lambda$  is missing, the upper or left parent is connected to the  $y$ . The  $M_1, \dots, M_m$  are the *legs* of  $y$ ; the  $N_0, \dots, N_i$  are the *arms* of  $y$ . A leg or arm that points to a  $\perp$  is mostly represented simply by a leg or arm pointing to empty space. This graphical representation makes the behaviour of game terms much better visible.

**Example:**



This is the representation of the term:

$$\lambda f g. \text{case}_1[g(\text{case}_1(g0\perp)1\perp)\perp][\text{case}_1(f(\lambda x. \text{case}_1 x01))22]\perp$$

of type  $((\iota \rightarrow \iota) \rightarrow \iota) \rightarrow (\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$ . It is a game term of *pregrade* 1. The output positions are the two positions of the number 2. If we replace the number 2 at the output positions by  $\perp$ , 0 or 1, then we get a game term of *grade* 1.

In the next subsection we develop a map  $\text{gt}_i^\sigma$  from finite terms to equivalent game terms such that  $M \prec N: \sigma$  entails  $\text{gt}_i^\sigma(M) \prec \text{gt}_j^\sigma(N)$ , where  $M, N$  are of grade  $i$  resp.  $j$ ,  $i \leq j$ . This means that the refutation of Berry's conjectures may be restricted to game terms. In the following subsection we extend our result to infinite game terms. They are needed for a full formulation of Berry's conjectures for first-order types (where they are valid).

**4.1. Finite Game Term Theorem.** We are given finite terms  $M \prec N$  and want to find equivalent game terms. First we must get rid of the  $Y$ s in the terms.

The map  $\omega: \text{PCF} \rightarrow \text{PCF}$  is taken from [5, 4] and called the *immediate syntactic value*:

$$\omega(M) = \begin{cases} \lambda x_1 \dots x_n. u \omega(M_1) \dots \omega(M_m), & \text{if } M = \lambda x_1 \dots x_n. u M_1 \dots M_m \\ & \text{with } u \text{ a variable or constant,} \\ & \text{i.e. } M \text{ is in head normal form} \\ \perp & \text{else} \end{cases}$$

Please note that in this definition  $u$  is not  $\perp$  or  $Y$ .

$\rightarrow_{\beta Y}$  is a one-step reduction with the  $\beta$ -rule or the rule  $YM \rightarrow M(YM)$  in any context.

As is known from [5, 4], if  $M \rightarrow_{\beta Y}^* N$ , then  $\omega(M) \prec \omega(N)$ .

**Lemma 4.2** (Approximation Lemma). *For every finite term  $M$  there is a term  $N'$  such that  $M \rightarrow_{\beta Y}^* N$  for some  $N$ ,  $N' \prec \omega(N)$ ,  $M \cong N'$  and  $N'$  is the  $\prec$ -least term with this property. This unique  $N'$  is called  $\text{approx}(M)$ .*

*Proof.* For the fully abstract cpo-model (and therefore for all f-models) the approximation continuity theorem [5, theorem 4.3.1] is valid:

$$\{ \llbracket \omega(N) \rrbracket \mid M \rightarrow_{\beta Y}^* N \} \rightarrow \llbracket M \rrbracket.$$

The set on the left is directed and  $M$  is finite, therefore there is  $N$  with  $M \rightarrow_{\beta Y}^* N$  and  $\llbracket M \rrbracket = \llbracket \omega(N) \rrbracket$ .

Now assume the type of  $M, N$  is  $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$ . Take any vector of closed terms  $A_1: \sigma_1, \dots, A_n: \sigma_n$  with  $\omega(N)A_1 \dots A_n \rightarrow^* m$  (integer constant).

By syntactic stability [4, theorem 2.8.8] [5, theorem 3.6.7] there is a  $\prec$ -least term  $N^* \prec \omega(N)$  with  $N^*A_1 \dots A_n \rightarrow^* m$ . Take as  $N'$  the  $\prec$ -lub of all these  $N^*$ .  $\square$

**Lemma 4.3.** *For all finite terms  $M \prec N$  it is  $\text{approx}(M) \prec \text{approx}(N)$ .*

*Proof.* Let  $M'$  be a term with  $M \rightarrow_{\beta Y}^* M'$  and  $M \cong \omega(M')$ .

As the  $\beta$ -rule and the  $Y$ -rule do not involve  $\perp$ , all these reductions  $M \rightarrow_{\beta Y}^* M'$  can also be done in  $N$ . (If  $A \prec B$  and  $A \rightarrow_{\beta Y} A'$ , then there is  $B'$  with  $B \rightarrow_{\beta Y} B'$  and  $A' \prec B'$ .) So there is  $N'$  with  $N \rightarrow_{\beta Y}^* N'$  and  $M' \prec N'$ , and of cause  $\omega(M') \prec \omega(N')$ .

By confluence of  $\rightarrow_{\beta Y}$  there is  $N''$  with  $N' \rightarrow_{\beta Y}^* N''$  and  $N \cong \omega(N'')$ .

It is  $\omega(N') \prec \omega(N'')$ , therefore  $\omega(M') \prec \omega(N'')$ .

$\text{approx}(M)$  is the least term  $X$  with  $X \prec \omega(N'')$  and  $M \sqsubseteq_{op} X$ .

$\text{approx}(N)$  fulfills the two conditions for  $X$ , therefore  $\text{approx}(M) \prec \text{approx}(N)$ .  $\square$

Now we have finite terms  $\text{approx}(M) \prec \text{approx}(N)$  without  $Y$ . The next step is to apply a  $\Psi_i^\sigma$ -like operator to the terms and reduce according to some reduction rules to game terms. The proof can be done in different ways:

In my first version I proved the termination of the reductions, formulated an invariant of the (eta-expanded) term structure, proved the invariance under the reductions and that they lead to game terms. This resulted in an induction on the reduction sequence, the induction step done by induction on the term, causing much rewriting bureaucracy. (This ugly proof will be available as supplementary material from my home page.)

Here we will see a more elegant proof based on an induction on the term from the beginning, with the aid of a reducibility predicate (see e.g. [19, theorem 3.1]). (Laird also uses a reducibility predicate to produce eta-expanded normal forms of a simply typed  $\lambda$ -calculus with lifting (without inconsistent values) [10, proposition 4.2].)

First we redefine the finite projection terms  $\Psi_i^\sigma$  as equivalent  $\text{case}_i$ -terms:

$$\Psi_i^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota} = \lambda f. \lambda x_1 \dots x_n. \text{case}_i[f(\Psi_i^{\sigma_1} x_1) \dots (\Psi_i^{\sigma_n} x_n)] 0 \dots i$$

To produce the game terms we define for every  $i \geq 0$  a big-step reduction relation  $M \downarrow_i N$  on  $\text{case}_i$ -terms. The mere existence of the game terms could be easily proved without  $\downarrow_i$ , but we want to give an explicit algorithm.

The values for  $\downarrow_i$

- of type  $\iota$ : the game terms of pregrade  $i$   
the terms of the form  $x A_1 \dots A_n$  with  $A_k$  game term of grade  $i$
- of type  $\sigma \rightarrow \tau$ : the game terms of grade  $i$   
the terms of the form  $x A_1 \dots A_n$  with  $A_k$  game term of grade  $i$   
suc, pre, if, if  $A_1$ , if  $A_1 A_2$  with  $A_1, A_2$  game terms of pregrade  $i$

As  $\downarrow_i$  will reduce under  $\lambda$ -abstractions, we need an auxiliary reduction relation  $\downarrow^\lambda$  that produces just a simple  $\lambda$ -abstraction by the following rules:

$$\lambda x. M \downarrow^\lambda \lambda x. M \qquad \frac{M \downarrow^\lambda \lambda x. M', \quad M'[x := N] \downarrow^\lambda \lambda y. P}{MN \downarrow^\lambda \lambda y. P}$$

Here are the rules for  $\downarrow_i$ . We abbreviate “is a game term of grade  $i$ ” by  $gi$  and “is a game term of pregrade  $i$ ” by  $pi$ . Generally the condition  $pi$  must be strengthened to  $gi$  if the produced whole term of the rule must be  $gi$ .

$V \downarrow_i V$  for all values  $V$

$$\begin{array}{c}
\frac{M \downarrow^\lambda \lambda x.M', \quad M'[x := N] \downarrow_i P}{MN \downarrow_i P} \qquad \frac{M \downarrow^\lambda \lambda x.M', \quad M' \downarrow_i M'' \text{ } gi}{M \downarrow_i \lambda x.M''} \qquad \frac{M \downarrow_i \perp}{MN \downarrow_i \perp} \\
\\
\frac{M \downarrow_i \text{ suc}, \quad A \downarrow_i A' \text{ } pi}{MA \downarrow_i A'[n := n+1]} \qquad \frac{M \downarrow_i \text{ pre}, \quad A \downarrow_i A' \text{ } pi}{MA \downarrow_i A'[0 := \perp, \quad n := n-1 \text{ for } n \geq 1]} \\
\\
\frac{M \downarrow_i \text{ if}, \quad A \downarrow_i A' \text{ } pi}{MA \downarrow_i \text{ if } A'} \qquad \frac{M \downarrow_i \text{ if } A', \quad B \downarrow_i B' \text{ } pi}{MB \downarrow_i \text{ if } A'B'} \qquad \frac{M \downarrow_i \text{ if } A'B', \quad C \downarrow_i C' \text{ } pi}{MC \downarrow_i A'[0 := B', \quad n := C' \text{ for } n \geq 1]} \\
\\
\frac{M \downarrow_i xA_1 \dots A_n (n \geq 0, \quad A_k \text{ } gi), \quad N_k \downarrow_i N'_k \text{ } pi, \text{ for } 0 \leq k \leq i}{\text{case}_i MN_0 \dots N_i \downarrow_i \text{case}_i(xA_1 \dots A_n)N'_0 \dots N'_i} \\
\\
\frac{M \downarrow_i A \text{ } pi, \quad N_k \downarrow_i N'_k \text{ } pi, \text{ for } 0 \leq k \leq i}{\text{case}_i MN_0 \dots N_i \downarrow_i A[n := N'_n \text{ for } 0 \leq n \leq i, \quad n := \perp \text{ else}]} \\
\\
\frac{M \downarrow_i xA_1 \dots A_n (n \geq 0, \quad A_k \text{ } gi), \quad N \downarrow_i A \text{ } gi}{MN \downarrow_i xA_1 \dots A_n A}
\end{array}$$

Now we come to the reducibility predicate. We pack all that we want to prove into its definition: the compatibility of the transformation with the order  $\prec$  and even the uniqueness of the reduction  $\downarrow_i$ .

**Definition 4.4** (reducibility predicate). Let  $i \leq j$ ,  $A$  a  $\text{case}_i$ -term and  $B$  a  $\text{case}_j$ -term of type  $\sigma = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$ ,  $n \geq 0$ .

$A \prec B: \sigma$  are  $(i, j)$ -transformable, written  $A \prec B: \sigma(i, j)$ , iff for all  $A_l \prec B_l: \sigma_l(i, j)$ ,  $1 \leq l \leq n$ , there are game terms  $A', B'$  of pregrade  $i$  resp.  $j$  with  $AA_1 \dots A_n \downarrow_i A'$  and  $BB_1 \dots B_n \downarrow_j B'$ ,  $A'$  and  $B'$  are unique for these reductions, and furthermore  $A' \prec B'$ .

Note that this definition does not take care of the free variables of  $A, B$ . Note also that it does not demand the *grade*  $i, j$  of  $A', B'$ , but the *pregrade*. So it will be applicable to general terms that do not restrict the integer constants, in the next lemma.

**Lemma 4.5.** Let  $A \prec B: \sigma$  be PCF-terms without  $\Upsilon$ . (Note: PCF-terms are without  $\text{case}_i$ .)

Let  $\{x_1^{\tau_1}, \dots, x_m^{\tau_m}\}$  be a superset of the free variables of  $B$ . For  $1 \leq k \leq m$  let  $A'_k \prec B'_k: \tau_k(i, j)$  be  $\text{case}_i$ - resp.  $\text{case}_j$ -terms that are  $(i, j)$ -transformable. Let  $s$  denote the substitution  $[x_1 := A'_1, \dots, x_m := A'_m]$  and  $t$  the substitution  $[x_1 := B'_1, \dots, x_m := B'_m]$ . Then  $As \prec Bt: \sigma(i, j)$ .

*Proof.* By induction on the term  $B$ .

**case**  $B = B_1 B_2$ :

First let  $A = A_1 A_2$ .

By the induction hypothesis we get  $A_1 s \prec B_1 t: \tau \rightarrow \sigma(i, j)$  and  $A_2 s \prec B_2 t: \tau(i, j)$  for some  $\tau$ .

Therefore  $(A_1 s)(A_2 s) \prec (B_1 t)(B_2 t): \sigma(i, j)$  and the proposition follows.

Now let  $A = \perp$ . By the same argument with  $A_1 = \perp$ ,  $A_2 = \perp$  we get  $\perp \perp \prec (B_1 t)(B_2 t): \sigma(i, j)$ .

By the  $\downarrow_i$ -rule for  $\perp$ -application the proposition follows.

**case**  $B = \lambda x.B^*: \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$ :

First let  $A = \lambda x.A^*$ .

Let  $A_l \prec B_l: \sigma_l(i, j)$  for  $1 \leq l \leq n$ .

By the induction hypothesis for  $B^*$  we get

$$A^*s[x := A_1] \prec B^*t[x := B_1]: \sigma_2 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota(i, j).$$

Therefore there are  $A', B'$  with  $(A^*s[x := A_1])A_2 \dots A_n \downarrow_i A'$  and  $(B^*t[x := B_1])B_2 \dots B_n \downarrow_j B'$ , with  $A', B'$  unique and  $A' \prec B'$ .

Then also  $(\lambda x. A^*s)A_1 \dots A_n \downarrow_i A'$  and  $(\lambda x. B^*t)B_1 \dots B_n \downarrow_j B'$  and  $A', B'$  are unique for this reduction.

Now let  $A = \perp$ . By the same argument with  $A^* = \perp$ ,  $A' = \perp$  and the rule for  $\perp$ -application we get the result.

**cases**  $B = x$  (variable),  $B = n$  (integer constant),  $B = \perp$  are clear.

For the case  $B = \perp$  and all subcases  $A = \perp$  the  $\downarrow_i$ -rule for  $\perp$ -application is needed.

**case**  $B = \text{if}$ :

First let  $A = \text{if}$ .

Let  $A_l \prec B_l: \iota(i, j)$  for  $1 \leq l \leq 3$ .

Then there are  $A_l \downarrow_i A'_l$  and  $B_l \downarrow_j B'_l$  ( $A'_l, B'_l$  unique) with  $A'_l \prec B'_l$ , for  $1 \leq l \leq 3$ .

Then if  $A_1A_2A_3 \downarrow_i A'_1[0 := A'_2, n := A'_3 \text{ for } n \geq 1]$

and if  $B_1B_2B_3 \downarrow_j B'_1[0 := B'_2, n := B'_3 \text{ for } n \geq 1]$ .

Both reductions are unique and the results are in relation  $\prec$ .

Now let  $A = \perp$ . We use the same argument. It is  $\perp A_1A_2A_3 \downarrow_i \perp$ .

**cases**  $B = \text{succ}$ ,  $B = \text{pre}$ : analogous to  $B = \text{if}$ . □

Next we prove a lemma that introduces the terms  $\Psi_i^\sigma$  into the transformation.

**Lemma 4.6.** *For all types  $\sigma = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$  the following three propositions are valid:*

(1) *For all  $A \prec B: \sigma(i, j)$  it is*

$$A(\Psi_i^{\sigma_1}x_1) \dots (\Psi_i^{\sigma_n}x_n) \prec B(\Psi_j^{\sigma_1}x_1) \dots (\Psi_j^{\sigma_n}x_n): \iota(i, j).$$

(2) *For all  $A \prec B: \sigma(i, j)$  there are  $A', B'$  with  $\Psi_i^\sigma A \downarrow_i A'$  and  $\Psi_j^\sigma B \downarrow_j B'$  such that both are unique for this reduction, and furthermore  $A' \prec B'$  and they are game terms of grade  $i$  resp.  $j$ .*

(3) *For all variables  $x^\sigma$  and  $i \leq j$ :  $\Psi_i^\sigma x^\sigma \prec \Psi_j^\sigma x^\sigma: \sigma(i, j)$ .*

*Proof.* By simultaneous induction on the type  $\sigma$ .

(1) By the induction hypothesis for (3) we get  $\Psi_i^{\sigma_k}x_k \prec \Psi_j^{\sigma_k}x_k: \sigma_k(i, j)$ , and the proposition follows.

(2) The proposition (1) means that there are game terms  $A'', B''$  with pregrade  $i$  resp.  $j$  such that  $A(\Psi_i^{\sigma_1}x_1) \dots (\Psi_i^{\sigma_n}x_n) \downarrow_i A''$  and  $B(\Psi_j^{\sigma_1}x_1) \dots (\Psi_j^{\sigma_n}x_n) \downarrow_j B''$ , with  $A'', B''$  unique for this reduction and  $A'' \prec B''$ .

Then  $\text{case}_i A''0 \dots i \downarrow_i A'''$  and  $\text{case}_j B''0 \dots j \downarrow_j B'''$ , where  $A'''$  is like  $A''$  with every number  $k$  at an output position replaced by the value of  $\text{case}_i k0 \dots i$ , analogous for  $B'''$ .

It is  $A''' \prec B'''$ , they are game terms of grade  $i$  resp.  $j$ .

It reduces

$$\begin{aligned} \Psi_i^\sigma A \downarrow^\lambda \lambda x_1 \dots x_n. \text{case}_i [A(\Psi_i^{\sigma_1}x_1) \dots (\Psi_i^{\sigma_n}x_n)] 0 \dots i \\ \Psi_j^\sigma B \downarrow^\lambda \lambda x_1 \dots x_n. \text{case}_j [B(\Psi_j^{\sigma_1}x_1) \dots (\Psi_j^{\sigma_n}x_n)] 0 \dots j. \end{aligned}$$

Then  $\Psi_i^\sigma A \downarrow_i A' = \lambda x_1 \dots x_n. A'''$  and  $\Psi_j^\sigma B \downarrow_j B' = \lambda x_1 \dots x_n. B'''$ . Both reductions are unique, it is  $A' \prec B'$  and they are game terms of grade  $i$  resp.  $j$ .



(3) It reduces

$$\begin{aligned}\Psi_i^\sigma x \downarrow^\lambda \lambda x_1 \dots x_n. \text{case}_i[x(\Psi_i^{\sigma_1} x_1) \dots (\Psi_i^{\sigma_n} x_n)]0 \dots i \\ \Psi_j^\sigma x \downarrow^\lambda \lambda x_1 \dots x_n. \text{case}_j[x(\Psi_j^{\sigma_1} x_1) \dots (\Psi_j^{\sigma_n} x_n)]0 \dots j.\end{aligned}$$

We have to prove that for all  $A_l \prec B_l: \sigma_l(i, j)$ ,  $1 \leq l \leq n$ , there are game terms  $A' \prec B'$  of pregrade  $i$  resp.  $j$  with  $(\Psi_i^\sigma x)A_1 \dots A_n \downarrow_i A'$  and  $(\Psi_j^\sigma x)B_1 \dots B_n \downarrow_j B'$  (with uniqueness of the reductions).

By the induction hypothesis of (2) for all  $l$  there are game terms  $A'_l \prec B'_l: \sigma_l$  of grade  $i$  resp.  $j$  with  $\Psi_i^{\sigma_l} A_l \downarrow_i A'_l$  and  $\Psi_j^{\sigma_l} B_l \downarrow_j B'_l$  (with uniqueness of the reductions). Therefore we get

$$\begin{aligned}(\Psi_i^\sigma x)A_1 \dots A_n \downarrow_i A' &= \text{case}_i[xA'_1 \dots A'_n]0 \dots i \\ (\Psi_j^\sigma x)B_1 \dots B_n \downarrow_j B' &= \text{case}_j[xB'_1 \dots B'_n]0 \dots j.\end{aligned}$$

These are even game terms of *grade*  $i$  resp.  $j$ . The reductions are unique.  $\square$

**Definition 4.7.** Let  $A$  be a  $\text{case}_i$ -term without  $\mathbf{Y}$  with  $A \prec A: \sigma(i, i)$ .

The unique game term  $A'$  of grade  $i$  with  $\Psi_i^\sigma A \downarrow_i A'$  is called  $\text{proj}_i^\sigma(A)$ .

For every finite term  $M: \sigma$  we get  $\text{approx}(M)$  without  $\mathbf{Y}$  with  $\text{approx}(M) \prec \text{approx}(M): \sigma(i, i)$  by lemma 4.5. (Note that finite terms are closed.)

We define the map  $\text{gt}_i^\sigma(M) = \text{proj}_i^\sigma(\text{approx}(M))$ , for  $M: \sigma$  finite term of grade  $i$ .

**Theorem 4.8** (Game Term Theorem). *If  $i \leq j$  and  $M \prec N: \sigma$  are finite PCF-terms of grade  $i$  resp.  $j$ , then  $\text{gt}_i^\sigma(M) \prec \text{gt}_j^\sigma(N)$  are game terms of grade  $i$  resp.  $j$  with  $M \cong \text{gt}_i^\sigma(M)$  and  $N \cong \text{gt}_j^\sigma(N)$ .*

*Proof.* By lemma 4.2 and 4.3 we get  $\text{approx}(M) \prec \text{approx}(N)$  without  $\mathbf{Y}$ . By lemma 4.5 it is  $\text{approx}(M) \prec \text{approx}(N): \sigma(i, j)$ . By lemma 4.6(2)  $\text{proj}_i^\sigma(\text{approx}(M)) \prec \text{proj}_j^\sigma(\text{approx}(N))$  are game terms of grade  $i$  resp.  $j$ . Furthermore  $M \cong \Psi_i^\sigma(\text{approx}(M)) \cong \text{proj}_i^\sigma(\text{approx}(M))$  and likewise for  $N$ .  $\square$

## 4.2. Infinite game terms.

**Definition 4.9.** An *infinite game term* of type  $\sigma$  is an ideal of game terms of type  $\sigma$  (of any grade), under the ordering  $\prec$ . (Infinite game terms can be construed as Böhm trees with infinite  $\text{case}$ -expressions, which we write as  $\text{case}_\infty MN_0N_1\dots$ ) The order  $\prec$  on infinite game terms is the subset order of the ideals. The semantics (in some f-model) of an infinite game term is the lub of the semantics of the members of its ideal, if the lub exists in the f-model.

**Definition 4.10.** Let  $M: \sigma$  be a closed PCF-term.

$\Psi_0^\sigma M \prec \Psi_1^\sigma M \prec \Psi_2^\sigma M \prec \dots$  is an ascending chain of finite terms with ascending grade. Define  $\text{gt}^\sigma(M)$  as the lub (in the order of infinite game terms) of the ascending chain of game terms  $\text{gt}_0^\sigma(\Psi_0^\sigma M) \prec \text{gt}_1^\sigma(\Psi_1^\sigma M) \prec \text{gt}_2^\sigma(\Psi_2^\sigma M) \prec \dots$

**Theorem 4.11** (Infinite Game Term Theorem). *If  $M \prec N: \sigma$  are closed PCF-terms, then  $\text{gt}^\sigma(M) \prec \text{gt}^\sigma(N)$  are infinite game terms with  $\llbracket M \rrbracket = \llbracket \text{gt}^\sigma(M) \rrbracket$  and  $\llbracket N \rrbracket = \llbracket \text{gt}^\sigma(N) \rrbracket$  in any f-model.*

*Proof.* By proposition 3.8 it is  $\llbracket \Psi_i^\sigma M \rrbracket \rightarrow \llbracket M \rrbracket$ , therefore  $\llbracket M \rrbracket = \llbracket \text{gt}^\sigma(M) \rrbracket$ , and likewise  $\llbracket N \rrbracket = \llbracket \text{gt}^\sigma(N) \rrbracket$ . As  $\text{gt}_i^\sigma(\Psi_i^\sigma M) \prec \text{gt}_i^\sigma(\Psi_i^\sigma N)$  for all  $i$ , we get  $\text{gt}^\sigma(M) \prec \text{gt}^\sigma(N)$ .  $\square$

## 5. THE SYNTACTIC ORDER IS NOT THE IMAGE OF THE STABLE ORDER

Berry's second conjecture in its finite form says that the stable order of the order-extensional fully abstract cpo-model of PCF (our greatest f-model) has the syntactic order as its image:

If  $a \leq b$  for finite  $a, b$  in the model, then there are normal form terms  $A, B$  with  $\llbracket A \rrbracket = a$ ,  $\llbracket B \rrbracket = b$  and  $A \prec B$ .

(The choice of the greatest f-model is not important, as all f-models coincide on their finite parts.)

In this section we will first show that Berry's second conjecture is valid in first-order types. Then we give our simplest counter-example in finitary PCF of second-order type, a chain of length 2. We also give examples of chains of any finite length. Finally we show that for every finite term there is a "universal" equivalent term that contains all possible syntactic parts.

For first-order types Berry's conjecture can be strengthened to the infinite case:

**Theorem 5.1** (Berry, Theorem 4.1.7 and 4.8.14 in [4]). *Let  $\sigma$  be a first-order type, and  $b \in D^\sigma$  in the greatest f-model. Then there is an infinite game term  $B$  with  $b = \llbracket B \rrbracket$ . Furthermore, for all such infinite game terms  $B$  and every subset  $t \subseteq \mathcal{T}(b)$  there is an infinite game term  $A \prec B$  with  $\mathcal{T}\llbracket A \rrbracket = t$ . (As infinite game term,  $A$  has a denotation in the greatest f-model.)*

*Proof.* Let  $\sigma = \iota \rightarrow \iota \rightarrow \dots \rightarrow \iota$  with  $n \geq 1$  arguments. In [4, 4.1.7] Berry shows that  $b \in D^\sigma$ , as the lub of a growing sequence of finite sequential functions, is itself sequential. Therefore: If  $b$  is not some constant function, then  $b$  is strict in some  $j$ -th argument. So  $B$  can be recursively constructed as infinite game term (with  $\text{case}_\infty$  the infinite case) in the form:

$$B = \lambda x_1 \dots x_n. \text{case}_\infty x_j B_1 B_2 \dots,$$

where  $B_i$  is a term with free variables  $x_1, \dots, x_{j-1} x_{j+1} \dots x_n$  for the residual function  $b_i$  given by

$$b_i x_1 \dots x_{j-1} x_{j+1} \dots x_n = b x_1 \dots x_{j-1} i x_{j+1} \dots x_n.$$

In [4, 4.8.14] Berry shows that  $A$  can be constructed in the same manner  $B$  was constructed, i.e. following the same choice of the variables for which the function is strict. We can describe the construction of  $A$  differently by using traces: The tokens of the trace  $\mathcal{T}\llbracket B \rrbracket$  correspond exactly to the branches of  $B$  that output a result, i.e. do not lead to  $\perp$ . We simply choose  $A \prec B$  by setting those branches of  $B$  that do not correspond to a token in  $t$  to the empty output  $\perp$ .  $\square$

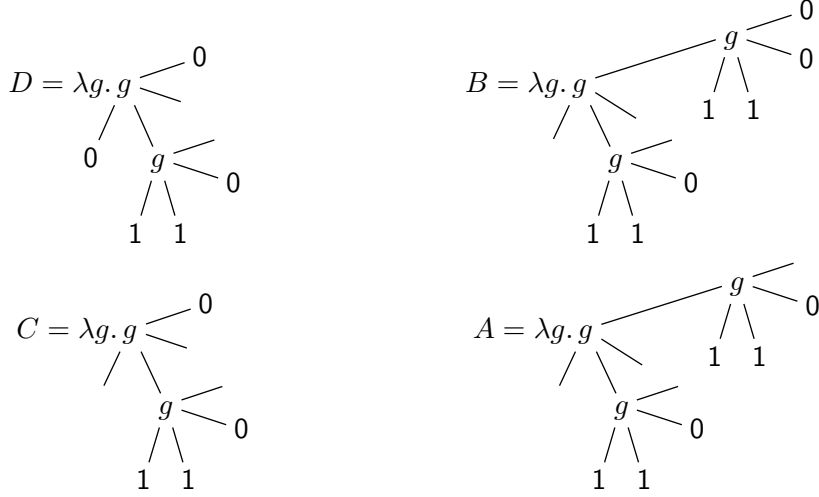
We conjecture that Berry's second conjecture is also true for second-order types with parameters of arity at most one:

**Conjecture 5.2.** Let  $\sigma = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$  with  $\sigma_i = \iota$  or  $\sigma_i = \iota \rightarrow \iota$  for all  $i$ . Let  $b \in \mathcal{F}_i^\sigma$  be a finite element of grade  $i$ .

Then there is a game term  $B$  of grade  $i$  for  $b$ ,  $b = \llbracket B \rrbracket$ , such that for every subset  $t \subseteq \mathcal{T}(b)$  that is secured in the sense of definition 2 of [7] there is  $A \prec B$  with  $\mathcal{T}\llbracket A \rrbracket = t$ . (Every semantic element is secured, so Berry's second conjecture would be fulfilled for these types.)

The proof of this conjecture is in preparation. It needs a new theory of (PCF-)terms that would exceed the frame of this paper.

**5.1. Refutation of Berry's second conjecture: A chain of least length 2.** Our simplest counter-example to Berry's second conjecture is in finitary PCF of second-order type  $(\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$ . We consider the following game terms  $A, B, C, D$ :



$$D = \lambda g. \text{case}_1(g \ 0 \ (\text{case}_1(g \ 1 \ 1) \perp 0)) \ 0 \perp$$

$$C = \lambda g. \text{case}_1(g \perp (\text{case}_1(g \ 1 \ 1) \perp 0)) \ 0 \perp$$

$$B = \lambda g. \text{case}_1(g \perp (\text{case}_1(g \ 1 \ 1) \perp 0)) (\text{case}_1(g \ 1 \ 1) \ 0 \ 0) \perp$$

$$A = \lambda g. \text{case}_1(g \perp (\text{case}_1(g \ 1 \ 1) \perp 0)) (\text{case}_1(g \ 1 \ 1) \perp 0) \perp$$

For illustration (not for the proof) we give the trace semantics of these terms:

$$\left. \begin{array}{l} A \left\{ \begin{array}{l} \{1 \ 1 \mapsto 1, \ \perp \ 0 \mapsto 0\} \mapsto 0 \\ \{ \perp \ 1 \mapsto 1, \ \perp \ 0 \mapsto 0\} \mapsto 0 \\ \{ \perp \ \perp \mapsto 0\} \mapsto 0 \\ \{ \perp \ 1 \mapsto 1, \ \ 0 \ 0 \mapsto 0\} \mapsto 0 \\ \{1 \ \perp \mapsto 1, \ \ 0 \ 0 \mapsto 0\} \mapsto 0 \\ \{ \perp \ 1 \mapsto 1, \ \ 0 \ 0 \mapsto 0\} \mapsto 0 \end{array} \right\} \\ B \cong C \end{array} \right\} D$$

We have  $A \prec B \cong C \prec D$ , therefore  $\llbracket A \rrbracket \leq \llbracket D \rrbracket$ . We will prove that this chain of two steps of  $\prec$  cannot be replaced by one single step.

Proof of the equivalence  $B \cong C$ : For any argument  $g$ , if  $Cg$  converges (i.e. reduces to an integer constant), then the subterm  $g11$  of  $C$  converges also. (There are only two possibilities for  $g$ : either  $\mathcal{T}(g) = \{\perp \perp \mapsto 0\}$ , or  $g$  demands its second argument.) Therefore it is possible to safely replace the result  $0$  in  $C$  by the term  $\text{case}_1(g11)00$ , i.e. to “lift”  $g11$  to the top level.

It is important to notice that this transformation cannot be performed with  $D$ : Here there are more possibilities for  $g$  to make  $Dg$  converge. It might be that  $\mathcal{T}(g) = \{0 \perp \mapsto 0\}$ , then the subterm  $g11$  does not converge.

The intuition of the example: We start with term  $D$ , working downwards step by step to  $A$  eliminating tokens of the trace. First the token  $(0 \perp \mapsto 0) \mapsto 0$  is eliminated getting  $C$

(and the other tokens with  $g$  demanding its first argument 0). Then it becomes possible to lift  $g11$ , we get  $B \cong C$ . Next we eliminate the token  $(\perp\perp\mapsto 0)\mapsto 0$  in  $B$  to get  $A$ . This is done by “forcing” the evaluation of the second argument of  $g$ , by demanding that  $g$  delivers different results for different arguments.

**Proposition 5.3.** *Let  $A, D$  be the game terms of grade 1 above. There are no game terms  $A', D'$  of grade 1 with  $A' \prec D'$  and  $A' \cong A$ ,  $D' \cong D$ . Then by the game term theorem 4.8 there are no PCF-terms  $A', D'$  with this property. Since we have seen that  $\llbracket A \rrbracket \leq \llbracket D \rrbracket$ , the proposition refutes Berry’s second conjecture.*

*Proof.* As game terms of grade 1,  $A'$  and  $D'$  should be of the form  $\lambda g.S$ , where  $S: \iota$  is a game term possibly with the only free variable  $g$ . We abbreviate  $S[g := M]$  as  $S[M]$ .

Let  $R, P, Q: \iota \rightarrow \iota \rightarrow \iota$  be the following terms:

$$\begin{aligned} R &= \lambda xy. \text{case}_1 y 0 (\text{case}_1 x \perp 1), & \mathcal{T}\llbracket R \rrbracket &= \{11 \mapsto 1, \perp 0 \mapsto 0\} \\ P &= \lambda xy. 0, & \mathcal{T}\llbracket P \rrbracket &= \{\perp \perp \mapsto 0\} \\ Q &= \lambda xy. \text{case}_1 x 0 \perp, & \mathcal{T}\llbracket Q \rrbracket &= \{0 \perp \mapsto 0\} \end{aligned}$$

We will prove: For any terms  $S, S'$  of the form above,

$$\text{if } S' \prec S \text{ and } S[Q] \rightarrow^* 0 \text{ and } S'[R] \rightarrow^* 0, \text{ then } S'[P] \rightarrow^* 0.$$

The proposition follows from this claim, as  $DQ \rightarrow^* 0$  and  $AR \rightarrow^* 0$ , but not  $AP \rightarrow^* 0$ .

The proof of the claim is by induction on the term  $S$ :

The cases  $S = \perp, 0, 1$  are clear.

Let  $S = \text{case}_1(gS_1S_2)S_3S_4$  and  $S' \prec S$  with  $S' = \text{case}_1(gS'_1S'_2)S'_3S'_4$ . (The remaining case  $S' = \perp$  is clear.)

Suppose  $S[Q] \rightarrow^* 0$  and  $S'[R] \rightarrow^* 0$ . Then  $S_1[Q] \rightarrow^* 0$ .

$R$  and  $Q$  are compatible in the Scott model of all continuous functions, the “parallel or” is an upper bound. Expressed differently,  $R$  and  $Q$  are compatible in the sense that they produce compatible integer results for the same argument. Therefore the semantics of  $S_1[R]$  and  $S_1[Q]$  must be compatible, so it is not possible that  $S_1[R] \rightarrow^* 1$ .

As  $S'_1 \prec S_1$ , it is also not possible that  $S'_1[R] \rightarrow^* 1$ .

Therefore  $(gS'_1S'_2)[R] \rightarrow^* 0$  (it must converge to get  $S'[R] \rightarrow^* 0$ ).

Hence  $S'[R] \rightarrow^* S'_3[R] \rightarrow^* 0$ .

On the other side we have  $S[Q] \rightarrow^* S_3[Q] \rightarrow^* 0$ .

Together we have  $S_3[Q] \rightarrow^* 0$  and  $S'_3[R] \rightarrow^* 0$ , and by the induction hypothesis for  $S_3$  follows:  $S'_3[P] \rightarrow^* 0$ .

Therefore  $S'[P] \rightarrow^* S'_3[P] \rightarrow^* 0$ .

Remark: As we base our proof on game terms, we gave a special induction hypothesis for the combination of  $\text{case}_1$  and  $g$ . The proof for general normal form terms is more complicated as it must work with  $\text{if}$  and  $g$  separately and use a more general induction hypothesis that takes the form of a Sieber sequential logical relation [23].  $\square$

**5.2. Chains of any length.** We have seen an example of a chain of two  $\prec$ -steps. Generally:

**Definition 5.4.** Let  $a \leq b$  be finite elements in an f-model.

A chain of length  $n \geq 1$  between  $a$  and  $b$  is a pair of sequences of terms  $(C_i), (D_i)$  with  $1 \leq i \leq n$  and  $a = \llbracket C_1 \rrbracket$ ,  $b = \llbracket D_n \rrbracket$  and  $C_i \prec D_i$ ,  $D_i \cong C_{i+1}$ .

If  $a = b$ , then we say there is a chain of length 0 between  $a$  and  $b$ .  
A chain is *of least length*  $n$  if there is no shorter chain.

By the game term theorem, if there is a chain of PCF-terms, then there is an equivalent chain of game terms.

Now we construct examples of chains of least length  $n + 1$  for any finite  $n \geq 0$ , by a sequential composition of  $n$  copies of our first example, each copy for a different argument  $g_i$ . For every  $n \geq 0$  let  $\sigma_n$  be the type  $(\iota \rightarrow \iota \rightarrow \iota) \rightarrow \dots \rightarrow (\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$  with  $n$  parameters. For every  $n$  we define two sequences of game terms  $C_n^i, D_n^i : \sigma_n$  with  $0 \leq i \leq n$ . First we define by induction on  $n$  the versions  $\bar{C}_n^i, \bar{D}_n^i$  without  $\lambda$ -binder:

$$\begin{array}{ll}
 \bar{D}_0^0 = 0 & \bar{D}_{n+1}^{n+1} = g_{n+1} \begin{array}{c} \bar{D}_n^n \\ / \quad \backslash \\ 0 \quad g_{n+1} \begin{array}{c} / \quad \backslash \\ 1 \quad 1 \end{array} \quad 0 \end{array} \quad \bar{D}_{n+1}^i = g_{n+1} \begin{array}{c} g_{n+1} \begin{array}{c} \bar{D}_n^n \\ / \quad \backslash \\ 1 \quad 1 \end{array} \quad 1 \quad 1 \\ / \quad \backslash \\ g_{n+1} \begin{array}{c} / \quad \backslash \\ 1 \quad 1 \end{array} \quad 0 \end{array} \quad \text{for } i \leq n \\
 \bar{C}_0^0 = \perp & \bar{C}_{n+1}^{n+1} = g_{n+1} \begin{array}{c} \bar{D}_n^n \\ / \quad \backslash \\ g_{n+1} \begin{array}{c} / \quad \backslash \\ 1 \quad 1 \end{array} \quad 0 \end{array} \quad \bar{C}_{n+1}^i = g_{n+1} \begin{array}{c} g_{n+1} \begin{array}{c} \bar{D}_n^n \\ / \quad \backslash \\ 1 \quad 1 \end{array} \quad 1 \quad 1 \\ / \quad \backslash \\ g_{n+1} \begin{array}{c} / \quad \backslash \\ 1 \quad 1 \end{array} \quad 0 \end{array} \quad \text{for } i \leq n
 \end{array}$$

We define  $C_n^i = \lambda g_n \dots g_1. \bar{C}_n^i$  and  $D_n^i = \lambda g_n \dots g_1. \bar{D}_n^i$ .  
For all  $n \geq 0, 0 \leq i \leq n$ :  $C_n^i \prec D_n^i$ . The proof is an easy induction on  $n$ .  
For all  $n \geq 1, i < n$ :  $D_n^i \cong C_{n+1}^{i+1}$ . Proof by induction on  $n$ :  
For  $n = 1, i = 0$  we have that  $D_1^0$  is the term  $B$ , and  $C_1^1$  the term  $C$  of our former example, both only with  $g$  replaced by  $g_1$ .  
For  $n := n + 1$ :  
For  $i = n$  we have  $D_{n+1}^n \cong C_{n+1}^{n+1}$  by the same argument as in our former example for  $B \cong C$ .  
For  $i < n$  we get  $D_{n+1}^i \cong C_{n+1}^{i+1}$  by the induction hypothesis.

All together for any  $n \geq 0$  we get a chain of length  $n + 1$  between  $\llbracket C_n^0 \rrbracket$  and  $\llbracket D_n^n \rrbracket$ :

$$C_n^0 \prec D_n^0 \cong C_n^1 \prec D_n^1 \dots D_n^{n-1} \cong C_n^n \prec D_n^n.$$

We want to prove that this chain has the least length.

First the intuition of the example: We use the terms  $R, P, Q$  of the proof of proposition 5.3 and name their traces:

$$r = \mathcal{T}[\llbracket R \rrbracket] = \{11 \mapsto 1, \perp 0 \mapsto 0\}, \quad p = \mathcal{T}[\llbracket P \rrbracket] = \{\perp \perp \mapsto 0\}, \quad q = \mathcal{T}[\llbracket Q \rrbracket] = \{0 \perp \mapsto 0\}$$

The trace of  $D_n^n$  contains all tokens  $p \dots pq \dots q \mapsto 0$ , with  $j$  arguments  $p$ ,  $0 \leq j \leq n$ . These tokens are in the upper branch of  $D_n^n$ . We work down from  $D_n^n$  eliminating all these tokens in  $n + 1$  steps.

In the  $j$ -th step ( $0 \leq j \leq n$ ) the token  $p \dots pq \dots q \mapsto 0$ , with  $j$  arguments  $p$ , is eliminated in  $D_n^{n-j}$ . (In  $D_n^{n-j}$  all the tokens of this form with less arguments  $p$  have already been

eliminated.) If  $j < n$  we proceed as follows: Following the upper branches in  $D_n^{n-j}$  we come to an occurrence of the variable  $g_{n-j}$ . It is the root of a subterm  $\bar{D}_{n-j}^{n-j}$ , its upper arm is  $\bar{D}_{n-j-1}^{n-j-1}$ . The elimination is by setting the first argument of this  $g_{n-j}$  to  $\perp$ , getting  $C_n^{n-j}$ . Only then it is possible to lift the lower  $g_{n-j}11$  to the top level, getting  $D_n^{n-j-1}$ . There the new  $g_{n-j}11$  at the top level gets two arms which are copies of  $\bar{D}_{n-j-1}^{n-j-1}$ . The lower arm (of these two) stays the same in the following transformations (it contains the token  $p \dots prq \dots q \mapsto 0$  with  $j$  arguments  $p$ ). The upper arm undergoes further eliminations of tokens  $p \dots pq \dots q \mapsto 0$ . These further eliminations are only possible after the separation of the two arms.

Finally in the  $n$ -th step the  $0$  which stands at the end of the upper branches of  $D_n^0$  is set to  $\perp$  getting  $C_n^0$ , eliminating the token  $p \dots p \mapsto 0$ .

**Proposition 5.5.** *Let  $n \geq 0$  and  $C_n^i, D_n^i$  be the terms defined above. Then the chain*

$$C_n^0 \prec D_n^0 \cong C_n^1 \prec D_n^1 \dots D_n^{n-1} \cong C_n^n \prec D_n^n$$

*between  $\llbracket C_n^0 \rrbracket$  and  $\llbracket D_n^n \rrbracket$  has the least length  $n + 1$ .*

*Proof.* We assume  $n \geq 1$  and suppose any chain between  $C_n^0$  and  $D_n^n$  and look at an intermediate  $\prec$ -step of this chain, i.e. we have the situation

$$C_n^0 \leq M \prec N \leq D_n^n.$$

We assume that some token of the form  $p \dots pq \dots q \mapsto 0$  is eliminated in this step. Let  $t$  be such token with the minimal number  $j$  of arguments  $p$ , and assume  $j < n$ .

Then we have

$$NP \dots PQ \dots Q \rightarrow^* 0, \text{ and } MP \dots PRQ \dots Q \rightarrow^* 0,$$

because  $C_n^0 \leq M$  (both with  $j$  arguments  $P$ ).

We can abstract the  $(j + 1)$ st argument in these terms and build the terms

$$N' = \lambda g. NP \dots PgQ \dots Q \text{ and } M' = \lambda g. MP \dots PgQ \dots Q.$$

It is  $M' \prec N'$ . We can transform  $M', N'$  to game terms and apply the argument in the proof of proposition 5.3 to deduce:  $M'P \rightarrow^* 0$ .

So  $MP \dots PPQ \dots Q \rightarrow^* 0$  (with  $j + 1$  arguments  $P$ ).

As  $Q \sqsubseteq_{op} P$ , we also have  $MP \dots PQ \dots Q \rightarrow^* 0$  for all  $k \geq j + 1$  arguments  $P$ .

All these arguments of  $M$  are minimal w.r.t. the stable order, because they are also minimal for  $D_n^n$  and it is  $M \leq D_n^n$ .

Therefore every token  $p \dots pq \dots q \mapsto 0$  with  $k \geq j + 1$  arguments  $p$  is in  $M$ .

This shows that from the tokens of the form  $p \dots pq \dots q \mapsto 0$  only the token  $t$  is eliminated in the step  $M \prec N$ . (For  $j = n$  this is trivially the case.) As there are  $n + 1$  of these tokens to be eliminated, the chain must have at least  $n + 1$  steps.  $\square$

Our example of a chain of least length  $n + 1$  has  $n$  functional parameters  $g_i$  of arity 2 and is of grade 1. We could transform it into an “equivalent” example with only one functional parameter  $g$  of arity 3 and terms of grade  $n$ , by coding  $g_i MN$  as  $giMN$ .

Our results suggest an improvement of Berry’s second conjecture:

**Conjecture 5.6** (Chain Conjecture). *If  $a \leq b$  are finite elements in an f-model, then there is a chain between  $a$  and  $b$ .*

We will refute also this conjecture in section 7.

**5.3. Universal terms.** Finally we show the existence of “universal” terms.

**Definition 5.7.** Let  $a$  be a finite element in an f-model. If  $b \leq a$  and there are terms  $A, B$  with  $a = \llbracket A \rrbracket$ ,  $b = \llbracket B \rrbracket$  and  $B \prec A$ , then  $b$  is called a *syntactic part* of  $a$ .

Let  $A$  be a closed term. All  $\llbracket B \rrbracket$  with  $B \prec A$  are called the *parts* of  $A$ .

$A$  is *universal* for  $a$  if  $\llbracket A \rrbracket = a$  and every syntactic part of  $a$  is a part of  $A$ .

In theorem 5.1,  $B$  was universal for  $b$ , even for infinite  $b$ .

**Proposition 5.8.** *Let  $a$  be a finite element. If  $b$  and  $c$  are syntactic parts of  $a$ , then  $b \sqcap c$  is a syntactic part of  $a$ .*

*Proof.* Let  $a = \llbracket B \rrbracket = \llbracket C \rrbracket$ ,  $B' \prec B$ ,  $b = \llbracket B' \rrbracket$ ,  $C' \prec C$ ,  $c = \llbracket C' \rrbracket$ .

Then we get  $a = \llbracket \inf BC \rrbracket$ ,  $b \sqcap c = \llbracket \inf B'C' \rrbracket$  and  $\inf B'C' \prec \inf BC$ .  $\square$

As we want the universal term to be a game term, we do not use the function term  $\inf$  but define an operation  $\text{glb}$  on game terms.

**Definition 5.9.** The operation  $\text{glb}$  on pairs of game terms of the same type is defined as:

$$\text{glb}(\perp, M) = \perp$$

$$\text{glb}(M, \perp) = \perp$$

$$\text{glb}(\lambda x_1 \dots x_n. M, \lambda x_1 \dots x_n. N) = \lambda x_1 \dots x_n. M[j := N[k := \perp \text{ for } k \neq j]]$$

where in the last case the variables of the terms are renamed so that the rule is applicable,  $M, N$  are of integer type and the brackets  $[\dots]$  enclose a substitution of the output positions of the game terms  $M$  resp.  $N$ , see the definition 4.1. (The output integers  $j$  of  $M$  are replaced by a term  $N'$ , where  $N'$  is  $N$  restricted to output integers  $j$ .)

It is  $\llbracket \text{glb}(M, N) \rrbracket = \llbracket M \rrbracket \sqcap \llbracket N \rrbracket$ . If  $M' \prec M$ ,  $N' \prec N$ , then  $\text{glb}(M', N') \prec \text{glb}(M, N)$ .

**Theorem 5.10** (universal game term). *Let  $a$  be a finite element of grade  $i$  in an f-model. Then there is a universal game term  $A$  of grade  $i$  for  $a$ .*

*Proof.* The set of syntactic parts of  $a$  is finite, because of property I of the domain w.r.t. the stable order. Let  $(a_j)$ ,  $1 \leq j \leq n$ , be an enumeration of them.

For every  $j$ , by the game term theorem there are game terms  $B_j, B'_j$  of grade  $i$  with  $a = \llbracket B_j \rrbracket$ ,  $a_j = \llbracket B'_j \rrbracket$  and  $B'_j \prec B_j$ .

We build the term  $A = \text{glb}(\dots \text{glb}(B_1, B_2) \dots B_n)$ .

For the syntactic part  $a_j$  of  $a$ , we build the term  $A'$  by replacing  $B_j$  in the expression for  $A$  by  $B'_j$ . It is  $A' \prec A$  and  $a_j = \llbracket A' \rrbracket$ .  $\square$

**Open Problem 5.11.** More properties of syntactic parts. Is the lub of two syntactic parts also a syntactic part? What about infinite terms and elements? Can the universal term be constructed without a priori knowledge of the syntactic parts?

## 6. THE STABLE ORDER IS NOT BOUNDED COMPLETE: NO BIDOMAIN

G rard Berry showed that the fully abstract order-extensional cpo-model of PCF (our greatest f-model) together with the stable order forms a bicpo, and conjectured that it is also a bidomain (Berry’s first conjecture). Here we repeat the definitions of both structures. We prove the conjecture for first-order types. Then we refute the general conjecture. Our first example is the stable lub of two finite elements for which the distributive law is not

valid. Our second example consists of two finite elements with stable upper bound but without stable lub. Both examples are in PCF of second-order type of grade 2.

**Definition 6.1** (Berry: 4.7.2 in [4]). A *bicpo* is a structure  $(D, \sqsubseteq, \leq, \perp)$  such that:

- (1) The structure  $(D, \sqsubseteq, \perp)$  is a cpo with least element  $\perp$  and with a continuous glb-function  $\sqcap$ .
- (2) The structure  $(D, \leq, \perp)$  is a cpo with least element  $\perp$  such that  $a \leq b \implies a \sqsubseteq b$  and for all  $\leq$ -directed sets  $S$  the two lubs are equal:  $\bigvee S = \bigsqcup S$ .
- (3) The function  $\sqcap$  is  $\leq$ -monotonic. (With (1) and (2) it follows that it is  $\leq$ -continuous.)
- (4) For all  $\leq$ -directed sets  $S$  and  $S'$ : If for all  $a \in S$ ,  $a' \in S'$  there are  $b \in S$ ,  $b' \in S'$  with  $a \sqsubseteq b$ ,  $a' \sqsubseteq b'$ ,  $b \leq b'$ , then  $\bigsqcup S \leq \bigsqcup S'$ .

In a bicpo: For all  $a \uparrow_{\leq} b$ ,  $a \sqcap b$  is also the glb w.r.t.  $\leq$ .

**Theorem 6.2** (Berry: 4.8.10 in [4]). *The domains  $(D^\sigma, \sqsubseteq, \leq, \perp)$  of the fully abstract order-extensional cpo-model of PCF are bicpos.*  $\square$

**Definition 6.3** (Berry: 4.4.10 in [4]). A cpo  $(D, \leq, \perp)$  is *distributive* if

- (1) it is bounded complete  
(This means that for  $a \uparrow_{\leq} b$  there is a lub  $a \vee b$ . And this entails with completeness that there is also a glb  $a \wedge b$  for all  $a, b$ , even for  $\leq$ -incompatible ones.)  
and
- (2) for all  $a, b, c \in D$  with  $b \uparrow_{\leq} c$ :  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ .

**Definition 6.4** (Berry: 4.7.9 in [4]). A bicpo  $(D, \sqsubseteq, \leq, \perp)$  is *distributive* if  $(D, \leq, \perp)$  is distributive and for all  $a \uparrow_{\leq} b$ :  $a \vee b$  is also the lub w.r.t.  $\sqsubseteq$ .

(Please note that in a distributive bicpo only for  $a \uparrow_{\leq} b$  it must be  $a \wedge b = a \sqcap b$ .)

**Definition 6.5** (Berry: 4.7.12 in [4]). A distributive bicpo  $(D, \sqsubseteq, \leq, \perp)$  is a *bidomain* if there is a  $\leq$ -growing sequence  $(\psi_i)_{i \geq 1}$  of finite projections w.r.t.  $\leq$  and with lub  $\bigvee \psi_i = \text{id}$ . (This means:  $\psi_i: D \rightarrow D$  is continuous w.r.t.  $\sqsubseteq$  and  $\leq$ ,  $\psi_i \leq \text{id}$ ,  $\psi_i \circ \psi_i = \psi_i$ ,  $\psi_i \leq \psi_{i+1}$ ,  $\psi_i(D)$  finite,  $\bigvee \psi_i = \text{id}$ .)

In this definition the sequence  $(\psi_i)$  is also a  $\sqsubseteq$ -growing sequence of finite projections w.r.t.  $\sqsubseteq$  and with lub  $\text{id}$ . Together with the the glb-function  $\sqcap$  it follows that  $(D, \sqsubseteq, \perp)$  is a Scott domain, a bounded complete  $\omega$ -algebraic cpo.

As we have explained in proposition 3.8 and 3.15, the conditions for  $(\psi_i)$  in the definition of bidomain are fulfilled for the fully abstract order-extensional cpo-model (and furthermore for all f-models) by the projections  $\psi_i^\sigma$ . In fact the  $(D^\sigma, \leq)$  are stable  $\omega$ -bifinite domains for the cpo-model, in the sense of definition 12.4.3 of [2].

To be precise, the condition of distributivity of the stable order was not conjectured by Berry in his thesis; there he remained agnostic. But in the state-of-the-art paper [5] we can read: “Unfortunately we are not able to show that the domains of the fully abstract model are bidomains, although we definitely believe it; the problem is to show that the  $\leq_{cm}$ -lubs are taken pointwise.”

First we clarify the situation for first-order types:

**Theorem 6.6.** *Let  $\sigma$  be a first-order type and  $(D^\sigma, \sqsubseteq, \leq, \perp)$  be the corresponding domain of any f-model.*

*The finite elements of  $D^\sigma$  fulfill distributivity w.r.t.  $\leq$  in  $D^\sigma$  in the following sense: For  $a, b \in \mathcal{F}^\sigma$  the glb in  $D^\sigma$  exists and is given by  $\mathcal{T}(a \wedge b) = \mathcal{T}(a) \cap \mathcal{T}(b)$ .*



For  $a, b \in \mathcal{F}^\sigma$  with  $a \uparrow \leq b$  the lub in  $D^\sigma$  exists and is given by  $\mathcal{T}(a \vee b) = \mathcal{T}(a) \cup \mathcal{T}(b)$ . It is taken pointwise and it is also the lub w.r.t.  $\sqsubseteq$ .

Then the distributive law is fulfilled by set theory on traces.

If  $D^\sigma$  contains a denotation for every infinite game term of type  $\sigma$  (this is the case for the game model and every greater f-model), then  $D^\sigma$  is the domain of the greatest f-model. In this case all elements  $a, b \in D^\sigma$  fulfill distributivity in the sense above. Therefore  $D^\sigma$  is a bidomain in this case.

*Proof.* Let  $a, b \in \mathcal{F}^\sigma$ .

We can apply theorem 5.1 and get a game term  $A$  with  $a = \llbracket A \rrbracket$ , and a game term  $C \prec A$  with  $\mathcal{T}[\llbracket C \rrbracket] = \mathcal{T}(a) \cap \mathcal{T}(b)$ . Define  $a \wedge b = \llbracket C \rrbracket$ ; it is finite and therefore in  $D^\sigma$ .

Now let  $a \uparrow \leq b$ , i.e. there is some  $d$  with  $a \leq d$  and  $b \leq d$ . By theorem 5.1 there are an infinite game term  $D$  with  $d = \llbracket D \rrbracket$ , and finite game terms  $A, B$  with  $a = \llbracket A \rrbracket$ ,  $b = \llbracket B \rrbracket$ ,  $A \prec D$ ,  $B \prec D$ . Take the syntactical lub  $E$  of  $A$  and  $B$ . It is  $\mathcal{T}[\llbracket E \rrbracket] = \mathcal{T}[\llbracket A \rrbracket] \cup \mathcal{T}[\llbracket B \rrbracket]$ , because in first-order game terms branches correspond to tokens. Define  $a \vee b = \llbracket E \rrbracket$ ; it is finite and therefore in  $D^\sigma$ . This lub is pointwise on the uncurried argument and therefore also the lub w.r.t.  $\sqsubseteq$ .

If  $D^\sigma$  contains a denotation for every infinite game term of type  $\sigma$ , then by theorem 5.1  $D^\sigma$  is exactly the domain of the greatest f-model. The construction of  $a \wedge b$  and  $a \vee b$  for any  $a, b \in D^\sigma$  is as above, only with infinite game terms.  $\square$

**Conjecture 6.7.** For all types of the form  $\sigma = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$ , with  $\sigma_i = \iota$  or  $\sigma_i = \iota \rightarrow \iota$ , Berry's first conjecture is valid, i.e.  $D^\sigma$  is a bidomain in the greatest f-model.

The proof of this conjecture is in preparation. It relies on the conjecture 5.2.

Now we prove some properties of stable upper bounds (sub) in f-models. (These are properties that are also valid in stable bifinite domains, see lemma 12.4.7 in [2].)

**Theorem 6.8.** Let  $D^\sigma$  be a domain of an f-model,  $\sigma = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \iota$ ,  $n \geq 0$ . Let  $X$  be a finite set of finite elements of  $D^\sigma$  that has a stable upper bound (sub) in  $D^\sigma$ . Let  $m$  be the maximal grade of the elements of  $X$ . For every sub  $x$  of  $X$  there is a unique minimal (w.r.t.  $\leq$ ) sub  $y$  of  $X$  with  $y \leq x$ . Every minimal sub of  $X$  is finite of grade  $m$ ; they are pairwise  $\leq$ -incompatible. The extensional lub  $\bigsqcup X$  is one of those.

*Proof.* Let  $x$  be a sub of  $X$ . Then the projection  $\psi_m^\sigma x$  is also a sub of  $X$ . Let  $Z$  be the set of all subs  $z$  of  $X$  with  $z \leq \psi_m^\sigma x$ ; it is a non-empty finite set of finite elements. Then  $y = \bigsqcap Z$  is the desired unique minimal sub of  $X$  with  $y \leq x$ .

Let  $a, b$  be two minimal subs of  $X$  that are  $\leq$ -compatible. Then  $a \sqcap b$  is also a sub of  $X$ , therefore  $a = b$ .

Let  $g = \bigsqcup X$  and  $h$  some sub of  $X$ . We have to show that  $f \leq g$  for every  $f \in X$ .

This is clear for  $n = 0$ , in the type  $\iota$ .

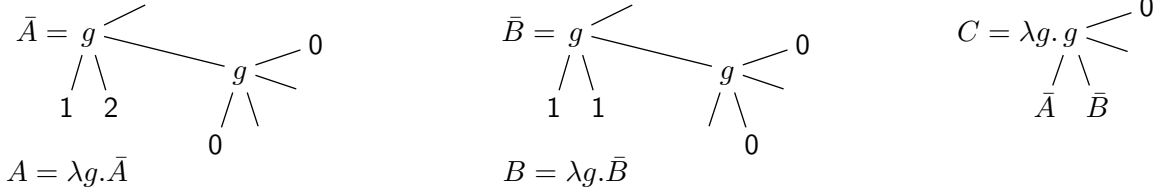
Now let  $n > 0$  and  $\vec{x}, \vec{y}$  be two vectors of arguments of type  $\sigma_1 \times \dots \times \sigma_n$  with  $\vec{x} \leq \vec{y}$ .

We have to show that  $f\vec{x} = f\vec{y} \sqcap g\vec{x}$ .

It is  $f\vec{x} = f\vec{y} \sqcap h\vec{x} \sqsupseteq f\vec{y} \sqcap g\vec{x}$ . And  $f\vec{x} \sqsubseteq f\vec{y} \sqcap g\vec{x}$  is clear.

This shows that  $g$  is a sub of  $X$ ; of course it is also minimal w.r.t.  $\leq$ .  $\square$

**6.1. A stable lub without distributivity.** Our first counter-example to Berry's first conjecture is of type  $(\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$  and of grade 2. We consider the following game terms  $A, B, C$ , where we use a  $\text{case}_1$  for a  $\text{case}_2$  with the third arm  $\perp$ :



Here are the traces of these terms:

$$\begin{array}{l}
 A \left\{ \begin{array}{l} \{0 \perp \mapsto 0, \ 1 \ 2 \mapsto 1\} \mapsto 0 \\ \{0 \perp \mapsto 0, \ 1 \perp \mapsto 1\} \mapsto 0 \end{array} \right\} \\
 B \left\{ \begin{array}{l} \{\perp 0 \mapsto 0, \ 1 \ 1 \mapsto 1\} \mapsto 0 \\ \{\perp 0 \mapsto 0, \ \perp 1 \mapsto 1\} \mapsto 0 \\ \{\perp \perp \mapsto 0 \quad \quad \quad \} \mapsto 0 \end{array} \right\} C
 \end{array}$$

It is  $A \leq C$  and  $B \leq C$ . We will show that  $C$  is the stable lub of  $A$  and  $B$ .

The intuition of the example:  $A$  and  $B$  do not contain the token  $\{\perp \perp \mapsto 0\} \mapsto 0$ , because their two occurrences of  $g$  are forced to evaluate their first resp. second argument, to get different results for different arguments. (This is the same trick that was used in the preceding section.)  $C$  adds to the tokens of  $A$  and  $B$  just the token  $\{\perp \perp \mapsto 0\} \mapsto 0$ , to separate  $\bar{A}$  and  $\bar{B}$ . (Note that a  $g$  for which  $Cg$  converges cannot demand both its arguments 00.) Therefore this lub does not fulfill distributivity. In  $C$  it is not possible to lift a differing term  $gMN$  to the top level that would eliminate that token, because the five occurrences of  $g$  in  $C$  cannot be “unified” to a common term that would always converge.

**Proposition 6.9.** *Let  $A, B, C$  be the game terms above.  $\llbracket C \rrbracket$  is the stable lub of  $a = \llbracket A \rrbracket$  and  $b = \llbracket B \rrbracket$ . Let  $d$  be the finite element with the trace  $\{\{\perp \perp \mapsto 0\} \mapsto 0\}$ . Then  $d \wedge (a \vee b) \neq (d \wedge a) \vee (d \wedge b)$ . This refutes Berry’s first conjecture.*

*Proof.* By the game term theorem 4.8 and the preceding theorem 6.8, every minimal sub of  $A$  and  $B$  can be represented by a game term of grade 2. Such a game term is of the form  $\lambda g. S$ , where  $S: \iota$  is a game term possibly with the only free variable  $g$ . We abbreviate  $S[g := M]$  as  $S[M]$ .

We use the following terms as arguments:

$$\begin{array}{ll}
 Q = \lambda xy. \text{case}_1 x 0 (\text{case}_2 y \perp \perp 1) & \mathcal{T}\llbracket Q \rrbracket = \{0 \perp \mapsto 0, \ 12 \mapsto 1\} \\
 R = \lambda xy. \text{case}_1 y 0 (\text{case}_1 x \perp 1) & \mathcal{T}\llbracket R \rrbracket = \{\perp 0 \mapsto 0, \ 11 \mapsto 1\} \\
 P = \lambda xy. 0 & \mathcal{T}\llbracket P \rrbracket = \{\perp \perp \mapsto 0\}
 \end{array}$$

$Q$  and  $R$  are compatible in the sense that they produce compatible results for the same argument. We will prove that for any term  $S$  of the form above:

$$\text{If } S[Q] \rightarrow^* 0 \text{ and } S[R] \rightarrow^* 0, \text{ then } S[P] \rightarrow^* 0.$$

The proof is by induction on the term  $S$ : The cases  $S = \perp, 0, 1, 2$  are clear.

Let  $S = \text{case}_2(g S_1 S_2) S_3 S_4 S_5$ .

For  $S[Q] \rightarrow^* 0$  it must be  $S_1[Q] \rightarrow^* 0$  or  $S_2[Q] \rightarrow^* 2$ .

(1) case  $S_1[Q] \rightarrow^* 0$ :

For  $S[R] \rightarrow^* 0$  it must be  $S_2[R] \rightarrow^* 0$  or  $S_1[R] \rightarrow^* 1$ .

(1.1) case  $S_2[R] \rightarrow^* 0$ :

We have  $S[Q] \rightarrow^* S_3[Q] \rightarrow^* 0$  and  $S[R] \rightarrow^* S_3[R] \rightarrow^* 0$ .

By the induction hypothesis for  $S_3$  we get  $S_3[P] \rightarrow^* 0$ , therefore  $S[P] \rightarrow^* 0$ .

(1.2) case  $S_1[R] \rightarrow^* 1$ :

This is not possible, as  $Q$  and  $R$  are compatible in the sense above.

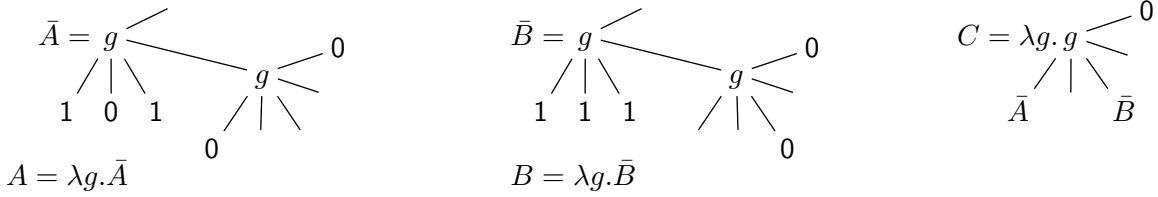
(2) case  $S_2[Q] \rightarrow^* 2$ :

For  $S[R] \rightarrow^* 0$  it must be  $S_2[R] \rightarrow^* 0$  or  $S_2[R] \rightarrow^* 1$ .

Both cases are not possible, as  $Q$  and  $R$  are compatible in the sense above.

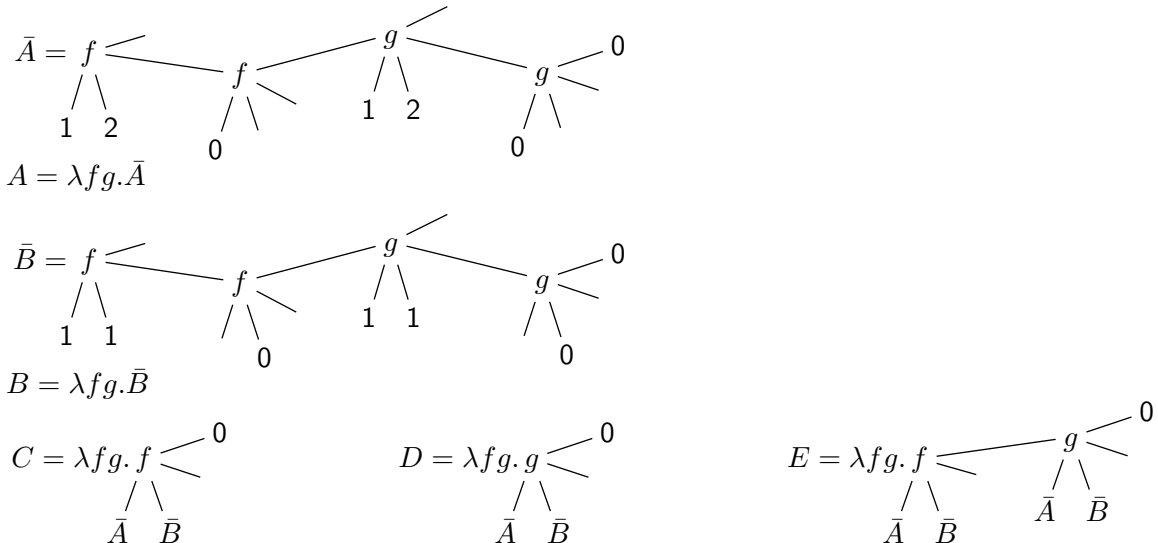
So we have shown that for every  $\sqsubseteq$ -upper bound  $D$  of grade 2 of  $A$  and  $B$  it must be  $DP \rightarrow^* 0$ . For a  $\leq$ -upper bound it cannot be  $D\perp \rightarrow^* 0$ . Therefore  $P$  is a  $\leq$ -minimal argument to fulfill  $DP \rightarrow^* 0$ . This means: Any minimal stable upper bound of  $A$  and  $B$  must contain the token  $(\perp\perp \mapsto 0) \mapsto 0$ . So  $C$  is the stable lub of  $A$  and  $B$ . (It is also the  $\sqsubseteq$ -lub.)  $\square$

Our counter-example is of grade 2 with  $g$  of arity 2. There is an “equivalent” example of grade 1 with  $g$  of arity 3:



**Conjecture 6.10.** In  $\mathcal{F}_1^{(\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota}$ , the finite elements of grade 1 of the type  $(\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$ , Berry’s first conjecture is valid; this subdomain is a bidomain. (This is a finite combinatorial problem and could be solved by a computer program.)

**6.2. Two elements without stable lub.** Now to our counter-example to bounded completeness of the stable order. It is of type  $(\iota \rightarrow \iota \rightarrow \iota) \rightarrow (\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$  and of grade 2. It employs the trick of our last example twice to two functional parameters. Consider the following game terms  $A, B, C, D, E$ , where we use a  $\text{case}_1$  for a  $\text{case}_2$  with the third arm  $\perp$ .



The traces of the terms are:

$$\begin{aligned}
\mathcal{T}[A] &= \{0 \perp \mapsto 0, 1 \ 2 \mapsto 1\} \mapsto \{0 \perp \mapsto 0, 1 \ 2 \mapsto 1\} \mapsto 0 \\
&\quad \perp \qquad \qquad \qquad \perp \\
\mathcal{T}[B] &= \{\perp 0 \mapsto 0, 1 \ 1 \mapsto 1\} \mapsto \{\perp 0 \mapsto 0, 1 \ 1 \mapsto 1\} \mapsto 0 \\
&\quad \perp \qquad \qquad \qquad \perp \\
\mathcal{T}[C] &= \mathcal{T}[A] \cup \mathcal{T}[B] \cup \{\{\perp \perp \mapsto 0\} \mapsto \perp \mapsto 0\} \\
\mathcal{T}[D] &= \mathcal{T}[A] \cup \mathcal{T}[B] \cup \{\perp \mapsto \{\perp \perp \mapsto 0\} \mapsto 0\} \\
\mathcal{T}[E] &= \mathcal{T}[A] \cup \mathcal{T}[B] \cup \{\{\perp \perp \mapsto 0\} \mapsto \{\perp \perp \mapsto 0\} \mapsto 0\}
\end{aligned}$$

In  $\mathcal{T}[A]$  and  $\mathcal{T}[B]$  the two indicated positions can be freely replaced by  $\perp$ , giving four combinations each.

$C, D, E$  are three stable upper bounds of  $A$  and  $B$ ; we will show that they are just the minimal stable upper bounds.  $E$  is the  $\sqsubseteq$ -lub of  $A$  and  $B$ .

The intuition of the example: In an upper bound of  $A$  and  $B$ , both have to be separated by some function call at the top level; because  $A$  and  $B$  cannot be “unified”. There are three ways to choose the separator:  $f$  or  $g$  or (both  $f$  and  $g$ ), realized by  $C, D, E$  resp.

**Proposition 6.11.** *Let  $A, B, C, D, E$  be the game terms above.  $\llbracket C \rrbracket, \llbracket D \rrbracket, \llbracket E \rrbracket$  are the minimal stable upper bounds of  $\llbracket A \rrbracket$  and  $\llbracket B \rrbracket$ . So  $\llbracket A \rrbracket$  and  $\llbracket B \rrbracket$  have no stable lub. (This again refutes Berry’s first conjecture.)*

*Proof.* By theorem 6.8, every minimal sub of  $A$  and  $B$  is of grade 2. By the game term theorem, we restrict to game terms of grade 2. These game terms must have the form  $\lambda f g. S$ . We use the terms  $Q, R, P$  of the proof of proposition 6.9. Our claim is: For every term  $S$  of the form above,

$$\text{if } S[f := Q, g := Q] \rightarrow^* 0 \text{ and } S[f := R, g := R] \rightarrow^* 0, \text{ then } S[f := P, g := P] \rightarrow^* 0.$$

The proof of the claim is by induction on the term  $S$  and follows exactly the proof of proposition 6.9. There is only one additional case  $S = \text{case}_2(f S_1 S_2) S_3 S_4 S_5$  of the same scheme.

So we have shown that for every  $\sqsubseteq$ -upper bound  $F$  of grade 2 of  $A$  and  $B$  it must be  $FPP \rightarrow^* 0$ . For a  $\leq$ -upper bound it cannot be  $F \perp \perp \rightarrow^* 0$ . Hence the minimal arguments to fulfill  $FPP \rightarrow^* 0$  must be  $(P, P)$ ,  $(P, \perp)$  or  $(\perp, P)$ . This is fulfilled by  $E, C, D$  respectively.  $\square$

## 7. REFUTATION OF THE CHAIN CONJECTURE

The chain conjecture 5.6 said that for finite elements  $a \leq b$  there is a chain between  $a$  and  $b$ , see the definition 5.4 of chain. We give here a counter-example in the type  $(\iota \rightarrow \iota \rightarrow \iota) \rightarrow (\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$  of grade 2. Consider the following game terms  $A, B$ :



$$\begin{array}{ll}
Q = \lambda xy. \text{case}_1 x 0 (\text{case}_2 y \perp \perp 1) & \mathcal{T}[[Q]] = \{0\perp \mapsto 0, \ 12 \mapsto 1\} \\
R = \lambda xy. \text{case}_1 y 0 (\text{case}_1 x \perp \perp 1) & \mathcal{T}[[R]] = \{\perp 0 \mapsto 0, \ 11 \mapsto 1\} \\
P = \lambda xy. 0 & \mathcal{T}[[P]] = \{\perp \perp \mapsto 0\}
\end{array}$$

We use the following terms as arguments for  $f$ :

$$\begin{aligned} Q' &= \lambda xy. \text{case}_1 x(\text{case}_1 y 0 \perp) 1 & \mathcal{T}[[Q']] &= \{00 \mapsto 0, \ 1\perp \mapsto 1\} \\ R' &= \lambda xy. \text{case}_1 y(\text{case}_1 x 0 \perp) 1 & \mathcal{T}[[R']] &= \{00 \mapsto 0, \ \perp 1 \mapsto 1\} \end{aligned}$$

The pairs  $(Q', Q)$  and  $(R', R)$  are compatible in the sense that their replacement into the same integer term leads to compatible results.

We will prove that for any terms  $S, S'$  of the form above:

If  $S' \prec S$  and  $S[\perp, P] \rightarrow^* 0$ ,  $S'[Q', Q] \rightarrow^* 0$ ,  $S'[R', R] \rightarrow^* 0$ , then  $S'[\perp, P] \rightarrow^* 0$ .

The proposition follows immediately from this claim.

The proof is by induction on the term  $S$ : The cases  $S = \perp, 0, 1, 2$  are clear.

Let  $S = \text{case}_2(gS_1S_2)S_3S_4S_5$  and  $S' \prec S$  with  $S' = \text{case}_2(gS'_1S'_2)S'_3S'_4S'_5$ . (The case  $S' = \perp$  is clear.)

Assume the three conditions of the claim.

Let  $(gS'_1S'_2)[Q', Q] \rightarrow^* q$  and  $(gS'_1S'_2)[R', R] \rightarrow^* r$ , both terms must converge to integer constants.

From the compatibility of  $(Q', Q)$  and  $(R', R)$  follows the compatibility of  $q$  and  $r$ , so either  $q = r = 0$  or  $q = r = 1$ .

$S'_2[Q', Q]$  and  $S'_2[R', R]$  are also compatible, so they cannot be 2 resp. 1.

Therefore  $q = r = 0$ .

Then we get  $S_3[\perp, P] \rightarrow^* 0$ ,  $S'_3[Q', Q] \rightarrow^* 0$ ,  $S'_3[R', R] \rightarrow^* 0$ .

By the induction hypothesis for  $S_3$  we conclude  $S'_3[\perp, P] \rightarrow^* 0$ , hence  $S'[\perp, P] \rightarrow^* 0$ . This fulfills the claim.

Now let  $S = \text{case}_2(fS_1S_2)S_3S_4S_5$ .

Then  $S[\perp, P] \cong \perp$ , so the claim is fulfilled.  $\square$

The refutation of the chain conjecture shows that already for second-order types the correspondence of stable and syntactic order is destroyed; there seems to be no simple syntactic characterization of the stable order. But certainly the two orders are related, but in which sense? A weaker conjecture that is now open is the following:

**Conjecture 7.2** (Maximality Conjecture). Every PCF-term that is syntactically maximal (i.e. contains no  $\perp$ ) is also stably maximal.

The existence of chains of any length suggests a kind of “metric” on finite elements  $a \leq b$ : If there is a chain between  $a$  and  $b$  of least length  $n$ , then the distance of  $a$  and  $b$  is  $n$ . If there is no chain, then the distance is  $\infty$ . But it might be doubted if this is meaningful, or if a transition  $A \leq B$  like the example above (without chain) should also be counted as some kind of elementary step of finite distance.

The example  $A \leq B$  above shows us that there are more syntactic means to produce a stably smaller term than replacing subterms by  $\perp$  (for the order  $\prec$ ). We can imagine that  $A$  is produced from  $B$  by “forcing” the upper  $g$  in  $B$  to demand one of its two legs, so that the token  $\perp \mapsto \{\perp \perp \mapsto 0\} \mapsto 0$  is eliminated.

We call this tentatively a “demand-forcing” step  $A \leq B$ . It might be possible to introduce new syntactic constructs to express such modifications of terms. This might lead to a new class of functionals that are generally not monotonic, but that still map extensionally compatible values to compatible results. This new class of functionals would be between the PCF-functionals and the sequentially realizable functionals of Longley [13].

But for our purposes only demand-forcing steps that stay in the realm of PCF would be interesting. We tentatively propose an *improved chain conjecture*: For all finite elements  $a \leq b$  there is a sequence  $(M_i)$  of terms with  $1 \leq i \leq n$ ,  $\llbracket M_1 \rrbracket = a$ ,  $\llbracket M_n \rrbracket = b$ , and for every  $i < n$  it is  $M_i \cong M_{i+1}$  or  $M_i \prec M_{i+1}$  or  $M_i \leq M_{i+1}$  by a demand-forcing step.

## 8. UNARY PCF

Here we will prove Berry's conjectures for unary PCF, with the aid of Jim Laird's results [10]. Unary PCF is the calculus of PCF without  $\mathbf{Y}$  and with the only constant  $0$  and  $\text{case}_0$ -expressions. Its semantics is given by the finite elements of  $\mathcal{F}_0^\sigma$  for all  $\sigma$ , with the orders  $\sqsubseteq$  and  $\leq$ .

We first repeat the general closure properties of the  $\mathcal{F}_i^\sigma$ , seen as embedded in the  $D^\sigma$  of an f-model, taken from lemma 3.3, proposition 3.15 and theorem 6.8.

**Proposition 8.1.** *The  $\mathcal{F}_i^\sigma$  are finite and downward closed w.r.t.  $\leq$ .*

*For  $a, b \in \mathcal{F}_i^\sigma$ ,  $a \sqcap b \in \mathcal{F}_i^\sigma$  is the glb w.r.t.  $\sqsubseteq$  in  $D^\sigma$  and  $\mathcal{F}_i^\sigma$ . For  $a \uparrow \leq b$  it is also the glb w.r.t.  $\leq$ .*

*For  $a, b \in \mathcal{F}_i^\sigma$  that are  $\sqsubseteq$ -bounded in  $D^\sigma$ ,  $a \sqcup b \in \mathcal{F}_i^\sigma$  is the lub w.r.t.  $\sqsubseteq$  in  $D^\sigma$  and  $\mathcal{F}_i^\sigma$ .*

*For a finite set  $X \subseteq \mathcal{F}_i^\sigma$  that has a stable upper bound, all minimal stable upper bounds of  $X$  are in  $\mathcal{F}_i^\sigma$ . The extensional lub  $\bigsqcup X$  is one of those. If  $X$  has a stable lub, then it is  $\bigsqcup X$ .*

To apply Laird's results on definable retractions, we augment unary PCF with product types  $\sigma \times \tau$ . The constructs of the whole language are:

$0: \iota, \perp^\sigma: \sigma, x^\sigma: \sigma$

If  $M: \tau$ , then  $\lambda x^\sigma.M: \sigma \rightarrow \tau$ .

If  $M: \sigma \rightarrow \tau$  and  $N: \sigma$ , then  $MN: \tau$ .

If  $M, N: \iota$ , then  $\text{case}_0 MN: \iota$ .

If  $M: \sigma$  and  $N: \tau$ , then  $\langle M, N \rangle: \sigma \times \tau$ .

If  $M: \sigma \times \tau$ , then  $\pi_1 M: \sigma$  and  $\pi_2 M: \tau$ .

The reduction rules are:

$(\lambda x.M)N \rightarrow M[x := N]$

$\text{case}_0 0M \rightarrow M$

$\pi_1 \langle M, N \rangle \rightarrow M$

$\pi_2 \langle M, N \rangle \rightarrow N$

This section needs the products only as auxiliary constructions for the first-order types that are the targets of Laird's retractions. In this section the underlying language is always the augmented unary PCF with products if products are not explicitly excluded.

Laird defines in [10] a categorical notion of *standard model* of unary PCF together with order-extensionality and partial extensional order at each type. He defines *parallel composition* as the function  $f$  with  $f\langle \perp, \perp \rangle = \perp$ ,  $f\langle \perp, 0 \rangle = f\langle 0, \perp \rangle = 0$ ,  $f\langle 0, 0 \rangle = 0$ . A model is *universal* at type  $\tau$  if every element of  $\tau$  is the denotation of a term.

**Definition 8.2** (Laird, definition 3.4 in [10]). Given types  $\sigma, \tau$ , a *definable retraction from  $\sigma$  to  $\tau$*  (in a model  $\mathcal{M}$ ) (written  $\text{Inj}: \sigma \leq \tau: \text{Proj}$  or just  $\sigma \leq \tau$ ) is a pair of (closed) terms  $\text{Inj}: \sigma \rightarrow \tau$  and  $\text{Proj}: \tau \rightarrow \sigma$  such that  $\llbracket \lambda x. \text{Proj}(\text{Inj } x) \rrbracket = \text{id}$  in  $\mathcal{M}$ .

**Lemma 8.3** (Laird, lemma 3.10 in [10]). *For any type  $\tau$  there is a natural number  $n$  such that there is a definable retraction from  $\tau$  to some binary product form of  $(\iota \rightarrow \iota)^n$ ; the same retraction for any standard order-extensional model without parallel composition.*

**Theorem 8.4** (Laird, theorem 3.11 in [10]). *Any standard model of unary PCF which is order-extensional and excludes parallel composition is universal.*

We can build the stable biorder model of unary PCF as a collection of bicpos  $(E^\sigma, \sqsubseteq, \leq)$  for every type  $\sigma$ : We start with  $E^\iota = \{\perp, 0\}$  and  $\perp \sqsubseteq 0, \perp \leq 0$ .

$E^{\sigma \times \tau} = E^\sigma \times E^\tau$  with the usual  $\sqsubseteq$  and  $\leq$ .

$E^{\sigma \rightarrow \tau}$  is the set of stable and monotone functions  $f: (E^\sigma, \sqsubseteq, \leq) \rightarrow (E^\tau, \sqsubseteq, \leq)$ . (If  $x \sqsubseteq y$  then  $fx \sqsubseteq fy$ . If  $x \leq y$  then  $fx \leq fy$ . If  $x \uparrow_\leq y$  then  $f(x \sqcap y) = fx \sqcap fy$ . Continuity conditions are not necessary as the domains are finite.)  $E^{\sigma \rightarrow \tau}$  is ordered by the usual  $\sqsubseteq$  and  $\leq$ .

$(E^\sigma, \sqsubseteq, \leq)$  is not only a bicpo, but a distributive bicpo where the stable lub of two  $\leq$ -compatible functions is defined pointwise, by proposition 4.7.10 in Berry's thesis [4]. (If  $f \uparrow_\leq f'$ , then  $(f \vee f')x = fx \vee f'x$ .) Therefore the stable lub of two elements is also defined by union on traces.

The stable biorder model fulfills the conditions of theorem 8.4, therefore it is universal (and fully abstract). This means that  $(E^\sigma, \sqsubseteq, \leq)$  is isomorphic to  $(\mathcal{F}_0^\sigma, \sqsubseteq, \leq)$  for types  $\sigma$  without products. In the following the semantics of unary PCF-terms is always taken in the model  $(E^\sigma, \sqsubseteq, \leq)$ . All this proves Berry's first conjecture for unary PCF:

**Theorem 8.5** (Laird [10]). *For every type  $\sigma$  without products, the structure  $(\mathcal{F}_0^\sigma, \sqsubseteq, \leq)$  is a distributive bicpo (hence also a bidomain as it is finite).*

*For  $a, b \in \mathcal{F}_0^\sigma$  with  $a \uparrow_\leq b$ ,  $a \vee b$  is given by  $\mathcal{T}(a \vee b) = \mathcal{T}(a) \cup \mathcal{T}(b)$  and this lub is taken pointwise for functions  $a, b$ .*

With the aid of Laird's definable retractions we can prove a strong form of Berry's second conjecture for unary PCF, based on the fact that it is valid for first-order types. First we need two lemmas on the reduction.

**Lemma 8.6.** *The reduction  $\rightarrow$  on unary PCF with products is confluent and strongly normalizing. Therefore it has unique normal forms. The normal form of a term of a type without products does not contain any product subterm.*

*Proof.* The confluence can be proved with the main theorem of [15], see also [6, theorem 10.4.15, page 576]: The rules of  $\rightarrow$  without the  $\beta$ -rule are confluent on the applicative terms (i.e. the terms without  $\lambda$ ), as they are orthogonal; they are left-linear and not variable-applying. Therefore their combination with the  $\beta$ -rule is confluent.

For the proof of strong normalization there seems to be no theorem in the literature that would provide an easy modular check for the simply typed  $\lambda$ -calculus with algebraic rewrite rules of our form.

Therefore we take the proof of strong normalization of the simply typed  $\lambda$ -calculus with products in the textbook [8, chapter 6] for the only atomic type  $\iota$  and augment it by the constant 0 and  $\text{case}_0$ -expressions. The proof stays literally the same. The only thing we have to add is a proof that if  $M, N$  are strongly normalizable, then  $\text{case}_0 MN$  is so; in the proof that all terms are reducible.  $\square$

**Lemma 8.7.** *Let  $\omega$  be the following map on unary PCF-terms (where  $n, m \geq 0$ ):*

$$\omega(\lambda x_1 \dots x_n. 0) = \lambda x_1 \dots x_n. 0$$



$$\begin{aligned}
\omega(\lambda x_1 \dots x_n. y M_1 \dots M_m) &= \lambda x_1 \dots x_n. y \omega(M_1) \dots \omega(M_m), \text{ for } y \text{ variable} \\
\omega(\lambda x_1 \dots x_n. \text{case}_0 MN) &= \lambda x_1 \dots x_n. \text{case}_0 \omega(M) \omega(N), \\
&\quad \text{if } \omega(M) = \text{case}_0 \dots \text{ or } \omega(M) = y \dots \text{ with } y \text{ variable} \\
\omega(\lambda x_1 \dots x_n. \langle M, N \rangle) &= \lambda x_1 \dots x_n. \langle \omega(M), \omega(N) \rangle \\
\omega(\lambda x_1 \dots x_n. \pi_1 M) &= \lambda x_1 \dots x_n. \pi_1 \omega(M), \text{ if } \omega(M) = y \dots \text{ with } y \text{ variable} \\
\omega(\lambda x_1 \dots x_n. \pi_2 M) &= \lambda x_1 \dots x_n. \pi_2 \omega(M), \text{ if } \omega(M) = y \dots \text{ with } y \text{ variable} \\
\omega(M) &= \perp, \text{ in all other cases}
\end{aligned}$$

$\omega(M)$  is a normal form prefix of  $M$ , it pushes  $\perp$ s upwards.

If  $M$  is a normal form, then  $\omega(M) \cong M$ .

If  $M \rightarrow^* N$ , then  $\omega(M) \prec \omega(N)$ .

If  $M \prec N$ , then  $\omega(M) \prec \omega(N)$ .

We define  $\text{nf}(M) = \omega(\text{the normal form of } M)$ .

For all  $M \prec N$  it is  $\text{nf}(M) \prec \text{nf}(N)$ .

*Proof.* The first four propositions are clear, we prove here the last one; the proof is similar to the one of lemma 4.3.

Let  $M', N'$  be the normal forms of  $M, N$ .

As the reduction rules for  $\rightarrow$  do not involve  $\perp$ , all the reductions  $M \rightarrow^* M'$  can also be done in  $N$ . (If  $A \prec B$  and  $A \rightarrow A'$ , then there is  $B'$  with  $B \rightarrow B'$  and  $A' \prec B'$ .)

So there is  $N''$  with  $N \rightarrow^* N''$  and  $M' \prec N''$ .

By confluence of  $\rightarrow$  it is  $N'' \rightarrow^* N'$ .

Then we get  $\text{nf}(M) = \omega(M') \prec \omega(N'') \prec \omega(N') = \text{nf}(N)$ . □

**Theorem 8.8.** For every type  $\sigma$  without products, for every  $a \in \mathcal{F}_0^\sigma$  there is a game term  $A: \sigma$  with  $a = \llbracket A \rrbracket$  such that for every  $b \leq a$  there is  $B \prec A$  with  $b = \llbracket B \rrbracket$ .

*Proof.* By Laird's lemma 8.3 there is a number  $n$  and a definable retraction  $\text{Inj}: \sigma \leq \tau: \text{Proj}$ , with  $\tau$  some binary product form of  $(\iota \rightarrow \iota)^n$ .

Let  $A'$  be a term for  $a$ ,  $\llbracket A' \rrbracket = a$ .

Let  $A'' = \text{nf}(\text{Proj}(\text{Inj } A'))$ .  $A''$  does not contain any subterm of product type.

By the game term theorem 4.8 we get the desired game term  $A = \text{gt}_0^\sigma(\Psi_0^\sigma A'')$  with  $A \cong A''$ , so  $\llbracket A \rrbracket = a$ .

Let  $C = \text{nf}(\text{Inj } A')$ .  $C = \langle C_1, \dots, C_n \rangle$  in some binary pair form, where  $C_i \cong \lambda x. \perp$  or  $\lambda x. 0$  or  $\lambda x. x$ .

Let  $b \leq a$ . Then  $\llbracket \text{Inj} \rrbracket b \leq \llbracket \text{Inj} \rrbracket a = \llbracket C \rrbracket$ .

For every  $i$ , if  $x \leq \llbracket C_i \rrbracket$  then  $x = \llbracket C_i \rrbracket$  or  $x = \perp$ . Therefore there is  $B' \prec C$  with  $\llbracket B' \rrbracket = \llbracket \text{Inj} \rrbracket b$ .

Let  $B'' = \text{nf}(\text{Proj } B')$ . It is  $A'' = \text{nf}(\text{Proj } C)$ . Therefore  $B'' \prec A''$ .

By the game term theorem 4.8 there is a game term  $B = \text{gt}_0^\sigma(\Psi_0^\sigma B'')$  with  $B \cong B''$  and  $B \prec A$ .

We have  $b = \llbracket \text{Proj} \rrbracket (\llbracket \text{Inj} \rrbracket b) = \llbracket \text{Proj} \rrbracket \llbracket B' \rrbracket = \llbracket B \rrbracket$ . □

**Remark:** Please note that Laird's retractions are incredibly intelligent, because they must introduce in the term  $A'' = \text{nf}(\text{Proj}(\text{Inj } A'))$  some nestings of variables that were not present in  $A'$ , to fulfill the proposition of the theorem.

It is a nice exercise (of three pages) to compute an example: Take  $\sigma = (\iota \rightarrow \iota \rightarrow \iota) \rightarrow \iota$  and  $A' = \lambda g. g00 : \sigma$ . The trace of  $A'$  is

$$\mathcal{T}[A'] = \{\{\perp\perp \mapsto 0\} \mapsto 0, \{0\perp \mapsto 0\} \mapsto 0, \{\perp 0 \mapsto 0\} \mapsto 0, \{00 \mapsto 0\} \mapsto 0\}.$$

Going through Laird's proof of lemma 8.3, we get complicated terms  $\text{Inj} : \sigma \leq \tau : \text{Proj}$  with  $\tau = (((\iota \rightarrow \iota) \times (\iota \rightarrow \iota)) \times (\iota \rightarrow \iota)) \times ((\iota \rightarrow \iota) \times \iota)$ .

We compute the normal forms:

$$\begin{aligned} \text{Inj } A' \rightarrow^* C &= \langle \langle \langle \lambda x.x, \lambda x.x \rangle, \lambda x.x \rangle, \langle \lambda x.x, \underline{0} \rangle \rangle \\ \text{Proj}(\text{Inj } A') \rightarrow^* A'' &= \lambda g. \text{case}_0[g(g0(g\underline{0}))0][g0(g\underline{0})] \end{aligned}$$

This term is much more expanded than needed.

If we replace the underlined  $\underline{0}$  in  $C$  by  $\perp$ , we get a term  $A''$  with both underlined  $\underline{0}$  replaced by  $\perp$ . The trace of this new term  $A''$  is  $\{\{\perp\perp \mapsto 0\} \mapsto 0, \{0\perp \mapsto 0\} \mapsto 0, \{\perp 0 \mapsto 0\} \mapsto 0\}$ . Note that there was no syntactically lesser term than  $A'$  with this trace.

**Remark:** Another recommended exercise for the reader is to encode our first counter-example (to Berry's second conjecture) of subsection 5.1 in unary PCF. The booleans are encoded by the type  $\beta = \iota \rightarrow \iota \rightarrow \iota$  as usual. The value 0 is represented by  $\lambda xy.x$ , 1 is represented by  $\lambda xy.y$ . There are three more inhabitants of  $\beta$ :  $\perp$ ,  $\lambda xy. \text{case}_0 xy$  and  $\lambda xy.0$ . The example is now of type  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ . The term  $D$  can be given an expanded form such that  $A \prec B \prec C \prec D$ . In  $D$  the top boolean  $\lambda xy.0$  is used (in one position) as the lub of  $\lambda xy.x$  and  $\lambda xy.y$ .

## 9. OUTLOOK

We have seen one trick to produce several examples which show that the stable order in PCF is not so regular as Berry had expected. These counter-examples have as necessary ingredients: at least two incompatible values and at least a second-order type with at least arity two of some functional parameter. To be precise, we still have to show that Berry's conjectures are valid in all second-order types with functional parameters of only arity one, see conjectures 6.7 and 5.2.

With the refutation of the chain conjecture in section 7 we have shown that there is no simple characterization of the stable order in terms of the syntactic order. In fact the counter-example shows that there is not only the syntactic order that causes the stable order, but that there are other syntactic relations that do it. Such another relation was identified as the relation of "demand-forcing", and an improved chain conjecture was tentatively proposed.

There should be some kind of full syntactic account of the stable order, at least for second-order types. For any type there should be syntactic conditions that are necessary for the relation  $A \leq B$  of terms. These should at least prove the maximality conjecture: Every PCF-term that is syntactically maximal is also stably maximal.

Another task is to further determine the category of domains in which the f-models live.

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